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Some Field Evidence relating to the Modes of Occurrence of Intrusive Rocks, with some Remarks upon the Origin of Eruptive Rocks in general. By J. G. Goodchild, of the Geological Survey, F.G.S., F.Z.S., Curator of the Collection of Scottish Mineralogy in the Edinburgh Museum of Science and Art. Communicated by R. H. Traquair, LL.D., M.D., F.R.S. Issued separately May 20, 1904, 197

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The People of the Faroes. (Oxon.) By Nelson Annandale, B.A. Communicated by Professor D. J. CUNNINGHAM, F.R.S. (MS. received Oct. 7, 1903. Read Nov. 2, 1903.)

PART I.—ANTHROPOMETRICAL.

The physical anthropology of the Faroes has recently been described in a very elaborate manner, as far as the island of Suderoe is concerned, by Dr F. Jørgensen (1), who was resident there as a medical man for some years. While pointing out, however, that the people of Suderoe differ considerably from those of the 'northern islands,' he only gives a comparatively small series of data regarding the latter, nor does he state to which of the northern islands the men he examined belonged, or even whether they came from one island or from several. Apart from Suderoe, there are sixteen inhabited islands (fig. 1) in the group, and between some of them very little communication exists even at the present day. In historical accounts of the Faroes the six following islands are usually called the 'northern isles,'—viz., Kalsoe, Kunoe, Boroe, Wideroe, Fugloe, and Svinoe,—but I take it that Dr Jørgensen would include at least Osteroe, Stromoe, and Waagoe also. His elaborate, laborious, and presumably accurate tables serve so well to point the moral that until a uniform method, a uniform standard, and a uniform set of anthropometrical instruments are adopted by anthropometrists of all nationalities final work in this branch of science will be impossible, that I have thought it well to put on record a small series of measurements taken by myself in the Faroes recently, and at the same time to point out wherein some of the data pretty generally adopted fail in accuracy, differing with the observer as well as the observed.

My measurements were taken in Thorshavn, the chief town in the islands, in August 1903, upon twenty adult males. The only value that can be claimed for so small a series is that it was obtained at a definite period and within a very limited area, for the men examined were all resident in the town. The length and breadth of the head, the length and breadth of the nose, the

length of the face, the bizygomatic and bigonial breadths, were taken with callipers of a simple type, while the height of the head, the auriculo-nasal and the auriculo-alveolar lengths were taken by means of Professor D. J. Cunningham's craniometer; all these measurements, therefore, were obtained directly, not by projections or estimations. The statures given can only be approximate, as all my subjects were measured with shoes or boots on their feet, and I was obliged to extract a varying number of millimetres in accordance with the kind of footgear worn.

The individuals measured are too few to make a rigid mathematical examination of the data regarding them legitimate, and they can give at best but an approximation to the race character of the people of Thorshavn. With so small a series perhaps the rough and ready method of examination by the aid of means and extremes is the best, as having the least appearance of finality.

The length of the head, as may be seen by the table, varies in the twenty adult men from 176 to 157 millimetres, while the mean of the series is 166, only .5 less than the mean of the two extremes. Though the extremes in the breadth of the head are less divergent from one another than those of the length, the mean is more divergent from that of the series, the former exceeding the latter by .9, and the variation is also greater. The mean index derived from these two measurements varies from 86.8 to 76.3; twelve of the men are brachycephalic, though four of these have an index between 80 and 81, while the remaining eight are mesaticephalic, only three being between 78 and 80; the mean, 80.6, is brachycephalic. If the skulls of these twenty men had been examined instead of their heads, it is probable that more than four would have been brachycephalic, and that would have been dolichocephalic; the mean index would certainly have been mesaticephalic. The mean cephalic index of Jørgensen's series of thirty-three men above the age of twenty-five is 80.4, and the extremes are 75.4 and 85.3; and the variation, as might be expected in a larger series, is slightly greater than in mine, while the difference between the mean of the series and that of mine, while the difference between the two series together, the mean is 80.7, and the mean of the extremes is 80.7, and the mean of the extremes is less. Taking

TABLE of Means and Extremes.
Twenty Men of Thorshavn.

		Gnathic Index.	
		Nasal Index.	
		Bigonial Index.	
		Facial Index.	
		Vertical Index.	
		Cephalic Index.	
		Auriculo-alveolar Length.	
		Auriculo-nasal Length.	
		Breadth of Nose.	
		Length of Nose.	
		Bigonial Breadth.	
		Bizygomatic Breadth.	
		Length of Face.	
		Height of Head.	
		Breadth of Head.	
		Length of Head.	
		Stature.	
Highest, .	176	202	min.
Lowest, .	157	185	172
Mean, .	166	192.6	155.2
			cm.
			136.4
			122.3
			111.8
			104.2
			35.7
			101.4
			80.62
			71.35
			91.9
			83.87
			65.66
			97.17

basion, but the limits within which this difference in level varies will be discussed later. At any rate, the thickness of the soft tissues of the scalp and the hair must quite compensate for it, if they do not cause the vertical height, taken as described, to be slightly greater, as is possible, than the true basi-bregmatic height. It is very unlikely, however, that in more than two cases at most would equal their parieto-squamosal breadth in the skull, and it is improbable that this would be found to be the case, could the skulls be measured, in a single instance. In the living men the mean breadth-height index of the head is 87.9, and the extremes are 98.6 and 77.9; the mean height is 136.4, and the extremes are 151 and 126 mm.

The length of the face, measured directly with the calliper between the bridge of the nose and the tip of chin, varies from 134 to 106 mm., with a mean of 122.3 mm., while the interzygomatic (or bizygomatic) breadth varies between 156 and 152 mm. in two cases out of twenty the length of the face is greater than the bizygomatic breadth, and in one the two measurements are equal. The complete facial index, calculated from these two measurements, varies from 101.8 to 77.9, and the man with the shortest face, which is considerably shorter than any other in the series, has the lowest index, though the man with the longest face, which is not so much longer than any other, has only the third index, the breadth being equal to the length. The measurements for the cephalic and vertical indices are easy to take with a fair degree of accuracy, and do not depend upon the play of the subject's features; but it is far otherwise with those for the facial index—an unfortunate fact, seeing that, provided all the measurements are taken by the same person, no index is of greater importance as a racial character. It makes all the difference in the world whether the length of the face is taken directly, or by projection from the vertex to the nasion and to the chin and by subsequent calculation, and it makes just as much difference whether the features of the subject are perfectly at rest or in any way distorted. I am not aware what manner exactly Dr Jørgen obtained what he calls the "longitudo naso-mentalis," or what degree of pressure he exerted in measuring his "latitudo bizygomatica."

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Length of Face.	
Height of Head.	
Breadth of Head.	
Length of Head.	
Stature.	
Highest, .	mm. 176 202
Lowest, .	mm. 157 185
Mean, .	mm. 166 192.6

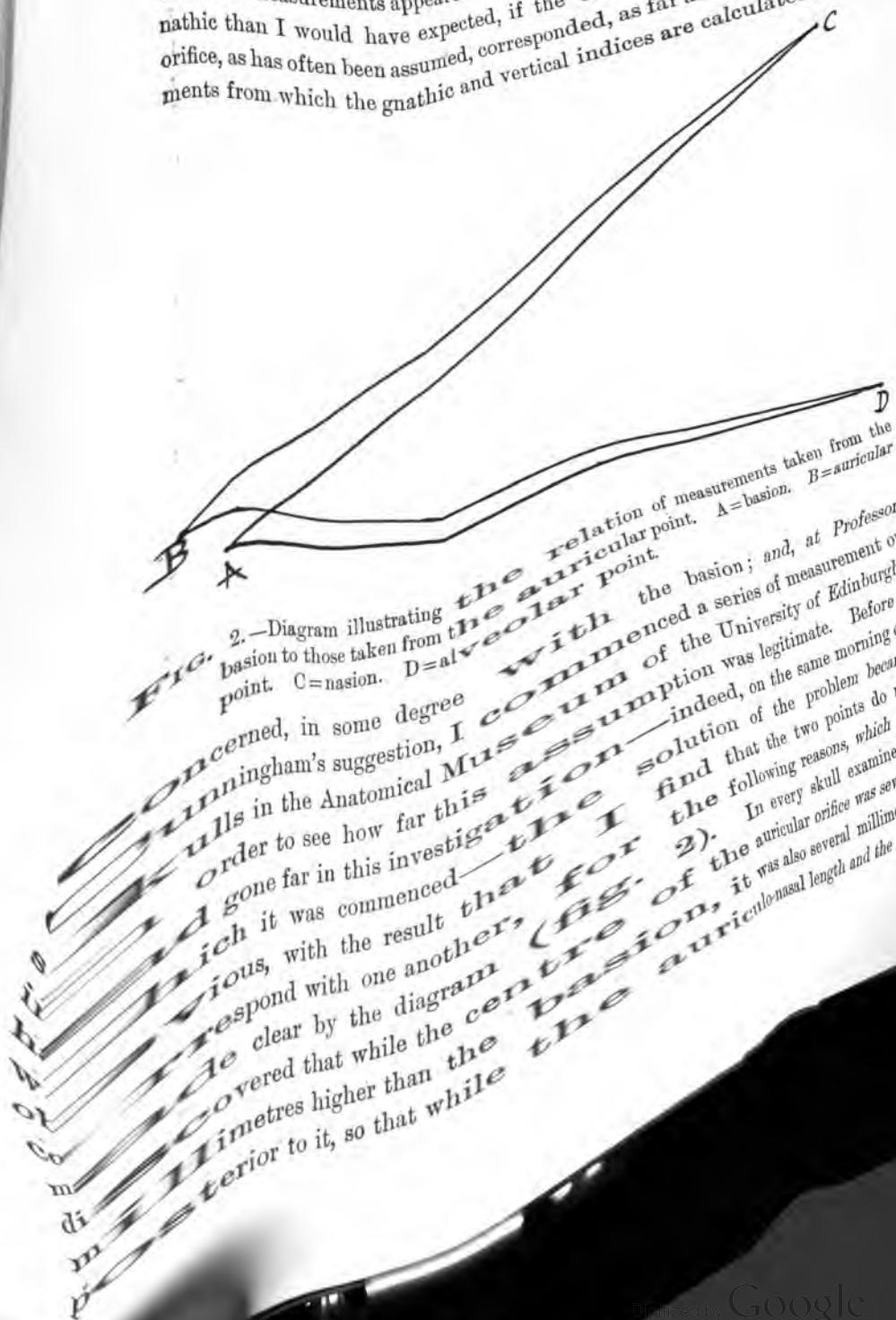
basion, but the limits within which this difference in level varies will be discussed later. At any rate, the thickness of the soft tissues of the scalp and the hair must quite compensate for it, if they do not cause the vertical height, taken as described, to be slightly greater, as is possible, than the true basi-bregmatic height. It is very unlikely, however, that in more than two cases at most the basi-bregmatic height of the individuals under discussion would equal their parieto-squamosal breadth in the skull, and it is improbable that this would be found to be the case, could the skulls be measured, in a single instance. In the living men the mean breadth-height index of the head is 87.9, and the extremes are 98.6 and 77.9; the mean height is 136.4, and the extremes are 151 and 126 mm.

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are often very prominent, and the face is frequently so flat, the eyes are so narrow, and the mouth is so big, that one is inclined to speculate as to the possibility of environment having induced some latent Mongoloid strain, inherited from prehistoric times, ere Iceland was colonised, to develop, or even whether environment alone could possibly have produced a similarity to the Esquimaux, not inherited at all. However, the time has not come to settle, or even to seriously discuss, such questions, and, in any case, they are beyond the point in dealing with the Faroemen, in whom there is little, if any, trace of any such phenomenon. All that can be said with reference to the point at issue is, that two observers who have examined the faces of the Faroemen get very different results with regard to the facial index, and that there is reason to believe that were a large number of Icelanders examined, they would be found to have considerably broader and flatter faces than the Faroemen.

The bigonial breadth is another measurement that depends very largely upon the individual observer, and probably has a very different relationship to the same measurement on the skull in different subjects. In taking it on the living person it is by no means easy to regulate the pressure exerted by the points of the callipers upon the soft tissues, and the degree or absence of such pressure makes a very great difference in the results obtained, while the extent to which the muscles which work the jaw are developed also influences the pressure in the points of the callipers as far as they will go without causing the subject pain. Personally, I now make it a practice to draw a more uniform standard of pressure in taking this measurement, and to individuals and as regards the callipers as far as possible. If this be done, the muscular development and the true breadth of the skull in my series, taken as described, is 111.8 mm.—21.6 mm. more than the mean bizygomatic breadth—and the extremes are 128 mm.

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the line joining the centre of this opening on one side of
the head to the same point on the other) and the bridge of the
nose, and the central point of the upper jaw between the two
central incisor teeth, respectively, the upper lip being lifted out
of the way in the latter measurement. The index calculated from
these two measurements appeared to make the people far more pro-
gnathic than I would have expected, if the centre of the auricular
orifice, as has often been assumed, corresponded, as far as the measure-
ments from which the gnathic and vertical indices are calculated are



mate, for all of them were taken, as mentioned above, on men who were not barefooted, and allowance had to be made for different kinds of footgear in different individuals; for these reasons I have only given the results in centimetres, though the measurements were originally taken in millimetres, and I believe that when recorded thus they are fairly accurate. The statures seem to fall into two very distinct series, those of 170 cm. and above and those below that figure; it is noteworthy that the last four men examined fall within the former category, showing how necessary a large series of measurements must always be in estimating the mean stature of a race. Dr Jørgensen's series of thirty-three men from the northern islands gives a mean of 169 cm., with extremes of 155 and 178 mm. Again, a very serious discrepancy exists between my measurements and his, for my mean is 166 cm., and my extremes are 157 and 176 cm., but I have not been able to discover whether his measurements were taken on barefooted subjects, or, if not, whether allowance was made for footgear. In any case, a visual inspection of the Faroemen makes it obvious that they are a very short race, perhaps as a result of in-breeding, though they are robust and well-built, and not, so far as I have been able to discover, degenerate in any other way. It is difficult, however, to discover, among them, as all bad cases but on the little island of Naalsoe, where several families, considering themselves to be descendants of the kings of Scotland, refused to marry the inhabitants of other islands, imbecility and total hereditary deafness are said to have been unusually common (3).

I have not thought it worth while to record my observations on the skin colour in detail, as I believe that this is due far more to the degree of exposure to which the individual has been subjected, climate, and even to altitude, than to race, at any rate within reasonable limits; for no amount of protection from the elements, cold, and no altitude would make a Negro white, or even give Italian the complexion of a Dane. All that can be said on this point as regards the dark hair have also a dark skin, which in some cases is as dark as that of an Italian, and that such persons have frequently

'old-fashioned' costume, which the men of the family delight to wear on festive occasions, is partly the result of their own imagination.

Having now dealt with the measurements and observations in my tables severally, I propose to inquire whether there are any obvious correlations between them, such as can be shown even so small a number of individuals as twenty. If we take the mean stature of the five tallest men, the mean stature of the five who come nearest to them, of the next five, and finally of the shortest, and if we take the mean of all the heads of the same five individuals in each of the four batches, we obtain the following results:—

	Stature.	Cephalic Index.	Vertical Index.	Facial Index.	Nasal Index.
Five tallest, . . .	173·4	82·0	70·4	94·5	63·0
Next five, . . .	167·4	81·1	72·0	88·1	68·1
Next five, . . .	163·4	79·8	70·9	90·1	66·0
Five shortest, . . .	159·8	79·3	70·6	94·4	64·0

As one figure is apt to throw out the mean in batches of five, we may further consider the head indices in from the point of view of the cephalic index, as the men are not those who have the five shortest indices.

	Cephalic Index.	Vertical Index.	Facial Index.
Five shortest heads, . . .	84·4	70·6	90·1
Next five, . . .	81·1	71·1	92·0
Next five, . . .	79·3	70·8	91·0
Five longest heads, . . .	77·2	69·8	89·0

From these tables it would appear that there is a close relationship between the stature and the cephalic index. Possibly, between the cephalic and the shape of the skull, as Sorgensen's data for Suderhoefer appear to show.

by their personal names, followed by those of their fathers with *dattir* added; men are for the most part referred to in the same manner, but with *sen* instead of *dattir*, while occasionally they adopt the name of their place of abode or birth instead of a patronymic. In the present list one man has a surname which has probably been introduced from southern Denmark or from the Schleswig-Holstein provinces, namely Djurhuus, while another has simply taken the name of his birthplace, Gjoueraa, a small village on the island of Stromoe, surnames being by no means a fixed institution in the country districts of the Faroes even at the present day, though they have gone far further in this direction than in Iceland. It is also worthy of note that a very large proportion of the names in my list are Biblical, and only a very small proportion Norse; while in a similar number of names from Iceland the majority would probably be found to be such as *Gisli*, *Herjolfur*, *Arni*, or the popular *Magnus*—a name introduced into Scandinavian countries through a misunderstanding of the latinized name of Charlemagne, a very popular hero in the ballads of the Faroes as in other Norse folk-lore.

The Faroes, we know, were colonised by vikings of Norse extraction, many of whom were descended from the Iberian chief-tains of the Hebrides and Ireland. There is no reason whatever to think that the islands had come, except perhaps occasionally, in records of their fellows in the way of finding 'solitudes.' There is good reason, however, to believe that Faroe, or, as it is properly derived, *Faeroe*, means 'sheep island,' though Landt (5) gives other derivations, and that the vikings found a breed of sheep without supposing either that the fact is difficult of explanation. His assumption be correct, the fact is established there; and if colonised by some race which had originally been sheep had disappeared, or else that the case with the "great", accidentally introduced by a wreck, as now quite extinct, having been to that of Soa in St Kilda, but islanders; it could hardly have come spontaneously into being on small islands separated by a very deep channel from any consider-

court named Eric and came with a great following to the Faroes. Naalsoe had been utterly depopulated by the Black Death, which raged in the islands at that date, and so the princess and her followers settled there. There she bore a son to Eric. Years later her father followed her, and when he came to Naalsoe he saw his grandson, whom he recognised because he was very like her, playing on the shore. Struck by the boy's beauty and manly appearance, he offered to forgive his daughter and her lover if they would return to Scotland with him. This they refused to do, remaining in the Faroes and having many other children there. The first-born son fell on a knife with which he was playing and killed himself, then the king of Denmark confiscated half the island from the princess because she was a Roman Catholic, but she and her other children, her followers and their descendants, peopled the island, and some of her descendants still refuse to marry outside the families who claim her as their ancestress. The present *amptmand* of the Faroes, the first native to be appointed to this position by the Danish Government, is of her kin. The whole story is, from the point of history, ridiculous, but I am inclined to agree with Robert Chambers (7), who heard the outlines of the tradition on a visit to the Faroes in the middle of last century, that in the main it may be true, any foreign lady of birth and wealth being easily transformed into a 'king's daughter' in a region so remote as the Faroes.

All these floating traditions, in any case, probably set forth a real fact, viz., that there was, subsequent to their original colonisation, a considerable influx of blood other than Norse into the Faroes; but whether the immigrants came as single individuals or in parties we cannot say with any more accuracy than we can give the date of their advent as late as 1874, the crowning of Denmark, shut off the Faroes from commerce with Iceland through extensive smuggling, and with the rest of Europe on the other; and of trade likely to lead to intermarriage. The fishermen of the Faroes met with fishing-smacks from Shetland on the high seas, and frequently hired themselves out to Shetland shipowners, learning to speak English from their mates, but they came home

water, so that they sank and were drowned. Mr Stanley Lane-Poole (9), moreover, in his *Barbary Corsairs*, states that Murād, a German renegade, "took three Algerine ships as far north as Denmark and Iceland, whence he carried off four hundred, some say eight hundred, captives . . . , and I have heard it stated in the Faroes that this expedition also visited these islands. Some years ago, while staying in the Westmann Isles, I took the trouble to translate the contemporary Icelandic accounts of Murād's raid, and of another, led by three Moorish captains, which also took place on the coast of Iceland in the same summer, that of 1627. These records (10) were collected and printed in Reykjavik about half a century ago. They contain no mention of a visit to the Faroes, and show that it is exceedingly improbable that any admixture of Algerian blood now exists even in Iceland. Between three and four hundred persons were taken prisoners by the two expeditions, and not more than forty, some of whom were women, got back to Iceland, the great majority being from the Westmann Isles, to which those who were ransomed by their friends or by the subscription raised for the purpose in Denmark returned. It is just possible that the women may have brought home with them children by Algerian masters, but it is exceedingly improbable that this would have been permitted; and even if they did, those who returned to the Westmann Isles, at any rate, have almost certainly left no descendants. After birth of *tetanus neonatorum*, behind them, for all children, almost without exception, who were born there died within a fortnight. Islands were constantly being repeopled from the north of Iceland, region which the corsairs did not visit (11, 12).

CONCLUSIONS.

My object, as regards the first part of this paper, has been critical rather than constructive, for I do not believe that measurements on the living person, even in series of considerable magnitude, can give more than a rough sketch of the physical condition of the islanders ascribe the recent extinction of this disease to the fact that new-born children were formerly laid on a mass of uncovered feathers that they are now placed on a covered mattress.

different Faroe villages, I see no reason to believe that the race is physically or mentally degenerate. A point which needs investigation even more urgently than the ethnology of the Faroes is the development of the Icelandic race, which has been more strictly isolated than the Faroemen, and in which some interesting peculiarities, I believe myself, might be discovered, even with so rough a method of examination as a large series of measurements of living individuals.

It only remains for me to express my thanks to Sir William Turner for his encouragement in the study of physical anthropology, and to Professor D. J. Cunningham, at whose suggestion the investigations embodied above were undertaken.

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(Issued 21st November 1908.)

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The binodal seiche, whose period is about 15.3 minutes, is usually very well marked. It is the commonest type, and lasts longer than the uninodal seiche. The node is probably somewhere in the neighbourhood of Inverfarigaig, but has not yet been accurately determined. It is also interesting because its period is less than half the period of the uninodal seiche, although, according to Du Boys, it ought always to be greater than half; and in most lochs it is so, the most notable exception being Lake Geneva. The basin of Loch Ness is so regular that it is difficult to explain it, as was attempted in the case of Lake Geneva, by assuming an oscillation of part of the loch. The period of the seiche is 8.8 minutes, is always of the same length, and is very regular. There are also oscillations of shorter period, which do not occur regularly, and which are of accuracy.

most lochs it is so, the most notable exception being the basin of Loch Ness, which is so regular that it is difficult to measure its length accurately. It was attempted in the case of Lake Geneva, by ascertaining the time of oscillation of part of the loch.

The polynodal seiche, whose period is 8.8 minutes, is always of small amplitude, but sometimes very regular. There are also oscillations of shorter period, but they do not occur regularly enough to allow of their measurement. On one or two occasions there were embroideries on the curve, which may have been due to transverse seiches. Owing to the narrowness of the loch, the period of such a seiche would only be about 1 minute. These embroideries may be due to a variety of causes, such as the wash of steamers, the opening of the lock gates in the canal, etc. It will only be possible to determine whether they are vibrations or transverse seiches by simultaneous observations at the two sides of the loch.

The range of atmospheric pressure at Fort Augustus included thunderstorms and earthquakes, but these had no very marked influence on the loch. It seems probable that the cause of seiches is sudden local variations of atmospheric pressure; and this view is supported by the records of a barograph at Fort Augustus. There are perhaps the uninodal and binodal seiches, and then strong, almost vertical gusts, and tidal oscillation.

regarding the causes of seiches, quantitative measurements of seiches will be done.

(Issued separately January 13, 1904.)

salmon being 51 lbs. On 6th July of this year (1903) seven bull trout were weighed together, and turned the scale at 214 lbs., showing the high average of 30 lbs. A small run of fish between 5 lbs. and 8 lbs. appeared with the grilse in July; and I may remark in passing that the Tay grilse are heavy as compared with the grilse of other rivers.

In general outline this so-called bull trout is in no way different from the shapely Tay salmon, and the appearance of the head, the outline of the gill cover, and shape of the preoperculum are identical. This is seen in Pl. fig. 1. The caudal fin also and the caudal peduncle are alike in like sizes of fish. The opportunity given me of viewing salmon interspersed with bull trout laid out in rows upon the sloping cement floor of the Tay Fisheries Co. Fish House at Perth enabled one not only to compare bull trout and salmon, but to note the variations which occur in both; and those variations I found to be in no way dissimilar.

The distinguishing feature of the bull trout is primarily one of surface marking. The *dorsum* is more or less thickly speckled with small black spots, and these are also to a varying extent displayed on the side, and the spots below the lateral line. A well-marked bull trout has the spots below the lateral line. But when one examines a large number of fish, examples are readily found with few spots; and one notices that a diminishing gradation which in no way differs from that seen in fish which are unquestionably pure salmon.

A peculiar characteristic of these fish, however, is the presence of 'maggots' (*Lerneopoda salmonae*, Linn.) on the gills, the parasite which commonly infests the *salmonae*, Linn. on the gills, the parasite *Lerneopodae* (*Lerneopthirus*) upon them to prove their comparative cleanliness, are nevertheless usually found in the sea into the river, and with tide I know of no other special infestation whereby this so-called salmon, and in my opinion no real structural difference exists.

A detailed examination reveals nothing in the dentition, fin-ray

No. 5. Female (Tay).
 Length $31'' \times 7''$ (79.2×17.8 cm.);
 weight 13 lbs.
 Length of head 14.3.
 Eye to post. of gill cover 8.2 cm.
 Teeth absent from vomer.
 Tail straight; caudal ped. 5.6 cm.
 Spots below lat. line to level of dorsal fin.
 Scales 12.
 Fin rays, D 12, V 9.
 Maggots, very few.

No. 6. Female (Tay).
 Length $28\frac{3}{4}'' \times 6\frac{1}{4}''$ (73.3×16 cm.);
 weight $10\frac{3}{4}$ lbs.
 Length of head 13.3 cm.
 Eye to post. of gill cover 7.8 cm.
 Teeth on head of vomer.
 Tail concave; caudal ped. 5.2 cm.
 Spots on shoulder below lat. line.
 Scales 12.
 Fin rays, D. 13, V 9, P 12.
 Maggots numerous.

This example had a *marked*
 salmon appearance.

No. 7. Female (Tay).
 Length $32'' \times 7''$ (81.7×17.8 cm.);
 weight $12\frac{3}{4}$ lbs.
 Length of head 15 cm.
 Eye to post. of gill cover 8.8 cm.
 One tooth on head of vomer.
 Tail straight; caudal ped. 5.8 cm.
 Spots below lat. line to level of post. margin of dorsal fin.
 Scales 12.
 Fin rays, D 13, P 12, V 9, A 10.
 Maggots, only two present.

No. 8. Female (Tay).
 Length $36\frac{1}{2}'' \times 7\frac{1}{2}''$ (93.2×19.2 cm.);
 weight $18\frac{1}{4}$ lbs.
 Length of head 17.7 cm.
 Eye to post. of gill cover 10 cm.
 One tooth on head of vomer.

In this series some fish were selected as having specially noticeable bull trout markings, while No. 6, when selected, gave rise to much discussion amongst the men present as to whether it

Spots, a small patch on shoulder only.
 Scales 11.
 Fin rays, D 13, P 12, V 9.
 Maggots not numerous.

No. 9. Female (from Loch Ness).
 Length $34\frac{1}{2}'' \times 8''$ (88×20.3 cm.);
 weight $19\frac{1}{2}$ lbs.
 Length of head 17 cm.
 Eye to post. of gill cover 10 cm.
 Teeth absent from vomer.
 Tail straight.
 Scales 12.

Fin rays, D 14, V 9.
 Spots all along dorsum and also below lat. line to level of front of dorsal fin.

Maggot, only one present.

No. 10. Female (from Loch Ness).
 Length $31'' \times 7''$ (79.2×17.8 cm.);
 weight $12\frac{1}{2}$ lbs.
 Length of head 15 cm.
 Eye to post. of gill cover 8.5 cm.
 Tail straight; caudal peduncle 5.8.
 Scales, 11 on right side, 10 on left, distinct.
 Fin rays, D 13, P 11, V 9.
 Spots below lat. line to level of dorsal fin.

Maggots absent.

No. 11 Female (Tay).
 Length $34\frac{1}{2}'' \times 7\frac{1}{2}''$ (88×19.7 cm.);
 weight $17\frac{1}{2}$ lbs.
 Length of head 17 cm.
 Eye to post. of gill cover 9.3 cm.
 Teeth absent from vomer.
 Scales 12.
 Fin rays D 14, V 9.
 Spots, very few below lat. line (1).
 Maggots numerous.

Had appearance of ill-conditioned salmon.

Others were selected as having specially noticeable bull trout markings, while No. 6, when selected, gave rise to much discussion amongst the men present as to whether it

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No. 8882	{ 24½ lbs. : 37¾": clean: ♂ : 14th Nov. 1902 : Westshot, Campsie.
No. 9005	{ 22½ lbs. : 40": clean: 13th Feb. 1903 : Flockie, in tidal water. 7½ lbs. : 27": unspawned grilse: ♀ : 22nd Nov. 1902 : Almond-mouth.
No. 9402	{ 12 lbs. : 31½": clean: 13th Aug. 1903 : Needle station. 4 lbs. : 25": grilse kelt: ♀ : 5th Feb. 1903 : Logierait, Upper Tay. 10½ lbs. : 30: clean: 31st July 1903: Flockie station.

The intervals of time are, in order, 556 days, 447 days, 191 days, 91 days, 295, and 176 days. In other words, we have one recapture after 18 months, and, at the other extreme, a recapture after only 3 months, but this latter is peculiar, since the fish was clean run when marked. It is just possible that this fish was 8882, may have been descending (without having spawned) when recaptured. The loss of weight is significant.

I have already noticed that the gill maggots are commonly *salmonae* is usually believed to be My attention was first called to the results obtained by the

after only 3 months, but this latter clean run when marked. It is just possible 8882, may have been descending (without having been recaptured. The loss of weight is significant.

I have already noticed that the gill maggots are commonly found on kelts. *Lerneopoda salmonae* is usually believed to be exclusively a fresh-water parasite. My attention was first called to the fact that this may not be the case in the results obtained by the marking of salmon marked in the Deveron Fishery Board for Scotland numbered 6508, was recaptured in the Cove, just south of Rivers Ugie, Ythan, Don, and recapture it had gained 2 $\frac{1}{2}$ lbs. in salt water, and between marking and recapture it had gained some time in Dee, is sufficient to show that the maggots were still attached to the gills when I received the salmon marking, Mr H. W. Johnston, who kindly associates himself with me in all the Tay marking. Many autumn fish with a few spawning fish are very commonly grilse proper the maggots are never so numerous as in 'bull trout,' or fish with certain fish. Our fresh trout markings, but I regard it as significant that fish marking experiments have shown

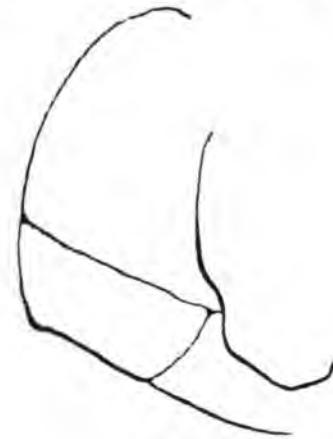
been a more appropriate title. This Tweed bull trout, otherwise known as the grey trout or round tail, is the *S. eriox*, as described by Yarrell, who, better I think than any other writer, seems to have recognised the rather distinct character of the fish. Günther refers to Yarrell's *S. eriox* under *S. cambricus*, the sewen, or English and Irish equivalent of our Scottish sea trout; and Day places the fish in the same category, with this difference, that he does not consider *cambricus* as specifically distinct from *trutta*.

Without entering at length into the wide question of the genealogy of migratory and non-migratory trout, it is advisable to recollect both the apparently great differences which exist between what I prefer to call local races of trout, and the infinite gradations which certainly exist to join such local races with one another and with the typical sea trout or the typical brown trout. The result of transporting brown trout eggs to New Zealand has shown how rapidly change of environment will produce a fish which our British Museum authorities diagnose as typical sea trout (*S. trutta*).

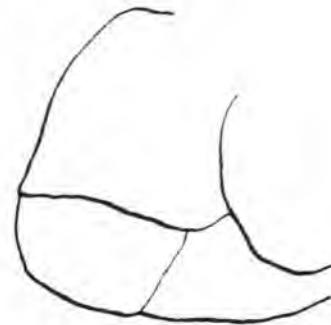
The late Sir James Maitland showed by different methods of feeding how Loch Leven trout could be made to resemble either *S. fario* or *S. trutta*; the beautifully silvery trout (*fario*) of some of our West Highland lochs, inaccessible to ascending fish; the characteristics of estuary trout, or, let us say, the creature usually described as *Salmo ferox*, are enough to show that either we must have a great many species, in accordance with the view adopted by Günther, or, laying stress on the intermediate gradations, we must regard all trout as belonging to one species, and that a plastic, and therefore perhaps a comparatively recent species. The name *S. trutta* was therefore perhaps a name adopted by Sir Humphrey Davy classed all our varieties under the name *S. eriox*; but it being maintained in 1878 that the term *eriox*, as applied to trout, was in reality the young of *S. salar*, gradual disentanglement of the naturalists, our present name in common use.

The typical gill cover I would represent thus:—

SKSS.



Tweed Trout.



S. trutta.

S. trutta.

Tweed Trout.

The general appearance of the head will be seen in the photographs of the male and female clean run fish (figs. 2 and 3). Relatively to the total length of the fish, I find that the head is contained from $4\frac{1}{2}$ to $5\frac{3}{4}$ times. The males examined in August varied from $4\frac{1}{2}$ to $4\frac{3}{4}$ times. The length in the females in each case had the head measurement $5\frac{3}{4}$ times. The caudal fin is also a well-marked feature. At a comparatively early age this tail fin becomes truncate or rounded at its outer margin. In *salar* and in *trutta* proper this never happens, so far as I am aware, except in females between 6 and 7 pounds. The female specimen photographed is well seen. An example weighing 7 lbs., and which was 18 $\frac{1}{2}$ inches long, was found to have the caudal fin slightly forked when fully extended. From this slightly forked condition in young fish the tail fin becomes first straight, then, with increased size and age, the rounded outer corner appears. Finally, in fish apparently so much thickened, the tail being not only rounded, but apparently as to have the free portions of the caudal fin rays noticeably short. All large specimens have not this appearance, but it is

I am indebted to Sir Richard Waldie Griffith, Bart., Chairman of the Tweed Commissioners, for specimens in spawning condition taken later in the year.

Though the Tweed trout cannot, in my opinion, be regarded as a species distinct from *trutta*, it is perhaps the best-defined variety of migratory trout in the British islands, and on this account might well, I think, retain the distinguishing name of *eriox*, in contradistinction to the variety *cambricus*. I am not familiar with the trout of the Coquet, but there seems no reason to doubt that the Tweed trout and the Coquet trout are of the same local race, and that Berwickshire and Northumberland form, as it were, the headquarters of the variety. Moreover, the history of the local fisheries seems to show that this variety has almost entirely superseded the sea trout proper. A point upon which more information is required is the relative distribution of this fish at the mouths of many of our Highland rivers, as referred to recently by Mr Harvie-Brown (*Fishing Gazette*, Oct. 10, 1903). In the Tweed, clean bull trout have been taken in January during netting for experimental purposes; and although the greatest runs are in early summer, and especially in late autumn, ^{the} ~~they~~ affect certain tributaries more than others, but push up to high spawning grounds.

In particulars of Estimated Annual Produce of the Fisheries of the River Tweed from 1808 to 1894, it appears that, whereas ^{at} the beginning of that period salmon or grilse, in process of trout were less numerous than either 1,033 trout. In 1844, the figures are 37,333 salmon, 25,324 grilse, and 8,535 trout. In 1884, there was a marked shrinkage, viz.—3271 salmon, 7817 grilse, and 18,003 grilse, and 99,256 trout. In 1894 we find the variety *eriox* now exhibits in which this trout has asserted itself more clearly to understand the well-defined character

(Issued separately January 30, 1904.)



lower part of the chest plays in effecting the emptying and (by resiliency) the consequent filling of the lungs. It has seemed desirable, therefore, to supplement them by further experiments, having for their object the exact determination of the amount of air exchanged, not only per respiratory movement, but also per unit of time, a factor which was left out of account in the earlier experiments, but one, nevertheless, of considerable importance.

The apparatus which was used in the experiments referred to in the report consisted of a counterpoised bell-jar, filled with air and inverted over water; to or from this the air of respiration was conducted from the mouthpiece (or mask) by a curved tube which passed through the water and opened into the bell-jar. When, therefore, air was drawn by the movement of inspiration from the bell-jar this sank in the water, and when air was forced into it by the movement of expiration it rose. These movements of the bell-jar were recorded and upon a slowly moving blackened cylinder, and the diameter and corresponding cubic contents of the bell-jar being known, the amount of air exchange was found by measuring the ordinates of the curves described on the cylinder. The readings, however, must be looked upon as only approximate, because, firstly, the bell-jar is cylindrical; and secondly, because the rapid movements imparted to it, a swing of its own which must have affected the record.

In order to obtain more accurate measure of the amount of air exchanged in respiration, these earlier experiments have been discarded, and we have used a carefully constructed graduated gasometer which was employed in the error which arises from the fact that the more a gasometer is raised out of the water in which it is inverted, the greater is the pressure exerted upon its contents. The air which is pumped out of the chest is alone measured, but it is clear that an equal amount must afterwards be passed in to take its place. The air is expired through either a mask or mouthpiece. In practice the latter is found to be more convenient. When it is used, the nostrils must be occluded to prevent accidental leakage.

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 respiratory movements; each 'rise' gives the amount of air expired; inspiration occurs during the 'tread' of each step; the intervals between the horizontal lines represent 500 c.c.; the time tracing shows a mark every ten seconds.

The tracings reproduced in figs. 3, 4 and 5 were all taken at the same time and from the same individual. The experiment begins in each case at the bottom, and is continued until the pen has nearly reached the top of the paper. The drum was then stopped and the cylinder (and pen) lowered (continuous vertical

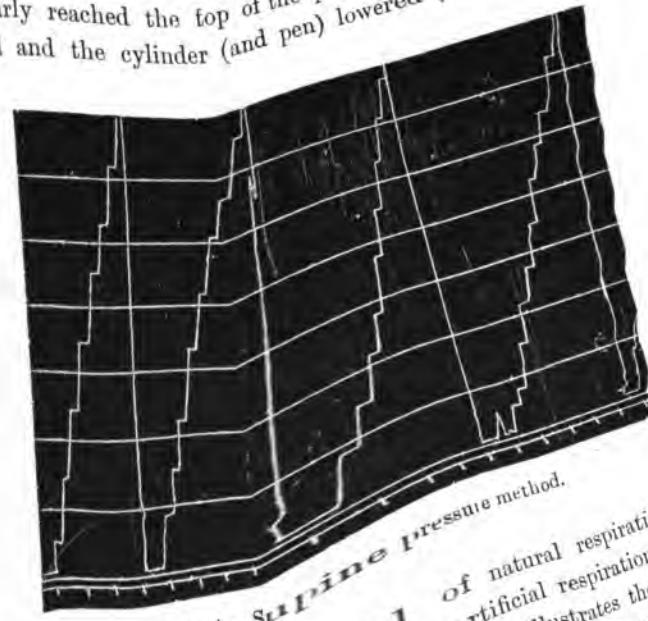


Fig. 4.—Supine pressure method

The following tables will serve to show the results yielded by the four principal methods which have been recommended for artificial respiration in man. In each case the respirations were performed during five minutes, but as the spirometer was only graduated to ten litres, it was necessary to take the amount of air yielded by each minute separately. In the intervals the subject was allowed to breathe naturally. There are also two tables (I. and II.) giving the amount of air breathed naturally into the spirometer, the circumstances being otherwise similar.

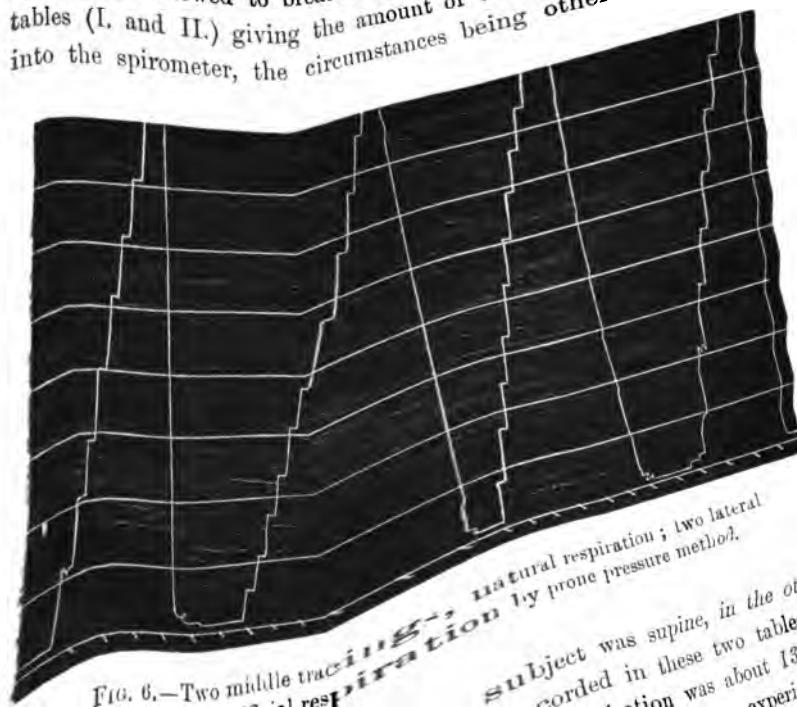


Fig. 6.—Two middle tracings, natural respiration; two lateral tracings, artificial respiration by prone pressure method.

the one series of these the subject was supine, in the other appeared that the normal rate recorded in these two tables, it was the rate aimed at in the conditions of respiration was about 13 per minute in the subject under the same conditions of the experiment. The amount of pressure produced by the operator by the weight of the upper part of the body of the subject was determined to be about 60 lbs. in Tables IV. and VI., was performing the artificial respirations, statistics of the subject of experiment are as follows:—

TABLE III.—*Silvester Method.* (Forcible traction upon the arms, followed by bringing of the arms back to the side of the chest and pressure upon the chest.)

	Number of Respirations.	Amount of Air in Cubic Cent.
1st minute,	13	3,700*
2nd "	12	2,100
3rd "	13	1,600
4th "	13	1,700
5th "	13	2,300
In 5 minutes,	64 respirations.	11,400 c.c. air exchanged.

Remarks.—The average number of respirations per minute was 12.8, and the amount of air exchanged per respiration averaged 178 c.c., and per minute 2280 c.c.

The amount of physical exertion required to effect even this amount of air exchange was very great, and it would have been impossible to continue it for any length of time. Moreover, the subject could scarcely sustain the effort not to breathe, for the amount of air he was receiving was quite inadequate, his natural tidal air being about 450 c.c. per respiration, and 5850 c.c. per minute (see Tables I. and II.). The subject was on the ground, with a folded coat under the shoulders; the operator at his head, in a semi-kneeling posture.

TABLE IV.—*Supine Pressure (Howard's Method.* (Intermittent pressure over the lower ribs, with the subject in the supine position.

	Number of Respirations.	Amount of Air in Cubic Cent.
1st minute,	14	4,000
2nd "	14	4,100
3rd "	14	3,900
4th "	13	3,500
In 5 minutes,	13	4,600
	64 respirations.	20,100 c.c. air exchanged.

The relatively large amount recorded here was probably due to the lungs having been unusually well filled when the experiment commenced.

pressure upon the back, the highest of these three numbers may be adopted, viz., 254 c.c. per respiration (3300 c.c. per minute). This amount, as compared with the tidal air of 450 c.c. per respiration, and 5850 c.c. per minute, is obviously inadequate; and, conformably with this, the subject experienced distinct distress towards the end of each minute, even when pressure was used. In the experiments without pressure, the minutes had to be cut up on this account into two periods of half a minute each.

Although not a great deal of physical exertion is required to roll a body half over in this way some 12 or 13 times a minute and alternately to press upon the back, yet the labour is much greater than that required by the simple pressure method. Such efficiency as the method may have depends largely upon the alternating pressure, for without this the rolling is quite ineffective. The reason why this pressure produces less effect than in the method next to be considered appears due to the fact that the time taken up by the rolling enables less time to be given to the pressure, so that this is almost necessarily inadequately performed if the normal rate of respiration is kept up.

TABLE VI.—*Prone Pressure Method.**—(This is similar to the Howard method (intermittent pressure on the lower ribs), but the subject is in the prone position.)

	Number of Respirations.	Amount of Air in Cubic Cent.
1st minute,	.	6,100
2nd "	12	6,800
3rd "	13	6,750
4th "	14	7,000
5th "	12	7,200
5 minutes,	14	
	65 respirations.	33,850

Remarks.—The rate of respiration was on the average 13, and the amount of air exchanged averaged 520 c.c. per respiration. This method is described in a paper communicated by the Royal Medical and Chirurgical Society, which was read on December 8th, 1903, and will be published in the *Med. Chir. Trans.*

of performing artificial respiration is that of *intermittent pressure upon the lower ribs with the subject in the prone position*. It is also the easiest to perform, requiring practically no exertion, as the weight of the operator's body produces the effect, and the swinging forwards and backwards some thirteen times a minute, which is alone required, is by no means fatiguing.* This statement also applies to the supine-pressure method when effected slowly and without undue violence. But not only is this method less efficient than the prone-pressure method, but there are undoubted dangers attending it, especially in those cases where the asphyxial condition is due to drowning. For in drowned individuals the liver is enormously swollen and congested, and ruptures easily, as Dr Herring and I found when endeavouring to resuscitate drowned dogs by this method of artificial respiration.† And further, the supine position is *contra-indicated* both in drowning and in asphyxia generally, since it involves the risk of obstruction of the pharynx by the falling back of the tongue, and vomited matter from the mouth and nostrils.

The Silvester method, in its favour. It has all the disadvantages of the supine position, is most laborious, and is Marshall Hall method, the relative inefficiency. As regards the exertion of pressure in the most effectual part of that method is quite unnecessary, and attending in the prone position; the rolling over is in addition to this method which is advocated by Bowles,‡ consisting in raising the one arm over the head after the body is placed in the lateral position, has been found, in measurements we have made, to introduce no serious change, but merely serves to augment the amount of air ex-

I have on one occasion continued it for nearly an hour without experiencing the least fatigue, and without any desire to breathe naturally or feeling at all inconvenienced. The Medical and Chirurgical Society, op. cit. Report of Committee of Royal Society for the Treatment of the apparently Drowned, London, 1903.

(Issued separately January 29, 1904.)

' a ' is small and also x , equation (ii) is really a particular case of equation (i); for we may put (i) in the form

$$\eta_s = 1 + x \log_e A + \frac{x^2 \log_e^2 A}{2!} + \frac{x^3 \log_e^3 A}{3!} + \dots$$

or, putting $\log_e A = a$

$$\eta_s = 1 + ax + \frac{a^2 x^2}{2!} + \frac{a^3 x^3}{3!} + \dots \quad (\text{iii}).$$

Considering aqueous solutions, we may (roughly) divide the dissolved substances into electrolytes and non-electrolytes. In the former class substances are known, e.g. potassium chloride, which do not follow the above formula (iii), but possess what may be called a 'negative' viscosity. Thus the viscosity of $\frac{1}{2}$ normal potassium chloride is less than that of water. Up to the present no non-electrolyte has been found to show this 'negative' viscosity. In the paper mentioned above, Rudorf drew attention to the fact that carbamide in dilute aqueous solution shows a 'negative' viscosity. I have repeated these measurements, and have also made determinations of the viscosity of acetamide in solution. These substances show a normal behaviour in their depression of the freezing-point.*

Carbamide (Urea).

Concentration.

$\frac{1}{10}$ mol. : : :
 $\frac{1}{4}$ " : : :
 $\frac{1}{2}$ mol. : : :
2 " : : :

	η_1	η_2	Δ
1.005	1.005	...	-0.001
1.012	1.011	...	-0.002
1.024	1.022	...	
1.015	1.045	...	
1.059	1.092	...	+0.03

Acetamide.

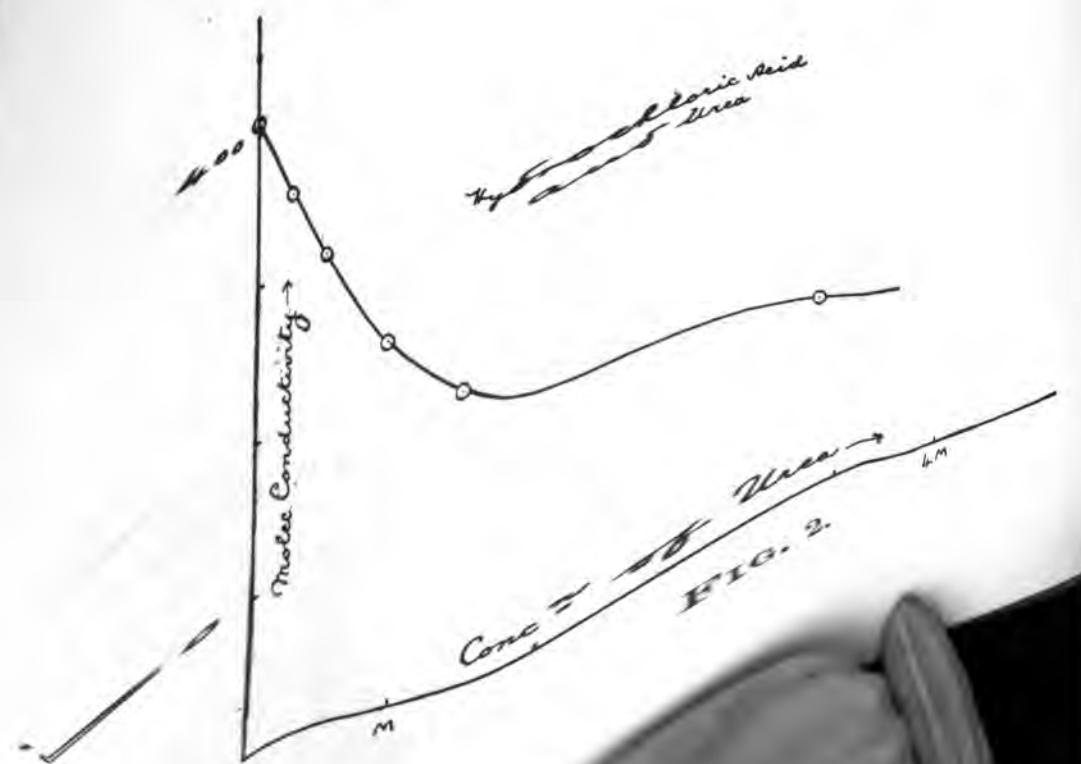
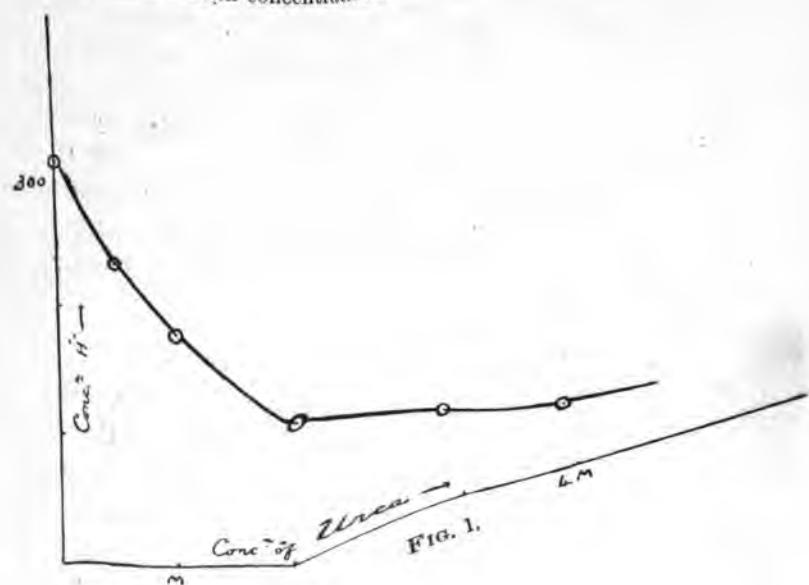
Concentration.

$\frac{1}{8}$ mol. : : :
 $\frac{1}{4}$ " : : :
 $\frac{1}{2}$ mol. : : :
2 " : : :

	η_1	η_2	Δ
1.013	1.014	...	+0.001
1.028	1.028	...	
1.057	1.057	...	
1.117	1.118	...	+0.001
1.250	1.250	...	

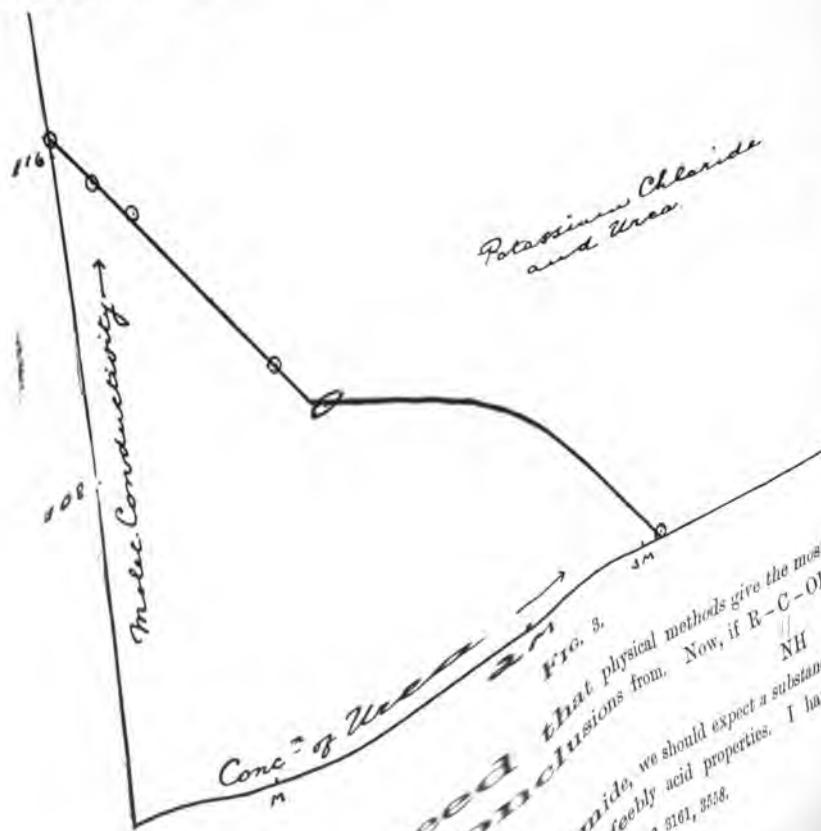
η_1 is the viscosity determined experimentally; η_2 that calculated from equation (iii). Δ is the difference of the calculated value from that observed; (Δ) in the case of carbamide being taken as -0.04, * *Zeitschrift für physikal. Chemie*, 2, 491 (1889).

I have represented these results in fig. 1.
The diminution in concentration of the H^+ ions may be observed



looked on as due to increased viscosity of the solution, as will be shown further on.

An amide is usually represented by the formula $R-C(=O)-NH_2$ where R stands for some radical. The formula $R-C(=O)-OH$ has also been suggested, although recent work * favours the adoption of the former. In investigating the constitution of such sub-



stances, it is generally agreed that physical methods give the most reliable results to draw conclusions from. Now, if $R-C(=O)-OH$ represented the formula of an amide, we should expect a substance of this kind to show at least ^{at least} ^{* Berl. Berichte, 36, 3142, 3161, 3558.} feeble acid properties. I have

1903-4.] Dr Munro on *Man in the Palaeolithic Period*. 115
observed in 1901 near the town of Stredne-Kolymsk, and an
expedition under Dr O. Hertz has recently transported the



FIG. 26.—Reindeer incised on wall of Combarelles.

entire animal in sections to Moscow, with the view of mounting it with its skin.



FIG. 27.—Figure of wild goat from the cave of Combarelles.

The total number of engravings in the cave of Combarelles, so far as they could be distinctly made out, is 109:—animals entire but not identified, 19; equidæ, 23; bovidæ, 3; bison, 2;

This plate represents an excellent picture of a bison (fig. 1) and a still more striking one of two reindeer (fig. 2). The original drawing of the former is painted in ochre, and measures 1 m. 50 in length and 1 m. 25 in height; that of the latter is 2 m. 10 in length and 1 m. 50 in height, and presents the peculiarity of having portion of the figure on the left executed in incised lines.

The total number of painted figures in this cave is 77:— aurochs, 49; indeterminate animals, 11; reindeer, 4; stag, 1; equidae, 2; antelopes, 3; mammoth, 2; geometrical ornaments, 3; scalariform signs, 2. The authors suggest that these paintings belong to a later period than the engravings on the walls of Combarelles, founding their opinion on the frequency of the figures of the bison, and the rarity of those of the reindeer and mammoth. Time will not allow me to enlarge on the details of these remarkable rock carvings and paintings, more than to say that MM. Capitan and Breuil have, by their explorations and published reports, greatly added to our knowledge of Palæolithic civilisation.

III. HUMAN CULTURE AND CIVILISATION IN THE PALÆOLITHIC PERIOD.

These illustrations, though only covering a small portion of the available materials, are sufficient to give a general idea of the salient features of the stage of culture to which the inhabitants of Europe had attained towards the close of the Palæolithic period. We have seen that all their works were characterised by a gradual development from simple to more complex forms. Implements, tools and weapons were slowly but surely being made more efficient, thus evincing on the part of their manufacturers a progressive knowledge of mechanical principles. Hence, French anthropologists have arranged these cave-remains in chronological sequence, using the names of the most typical stations to define various stages of culture, as *Moustérien*, *Solutréen*, and *Magdalénien*. The earliest troglodytic station, according to the classification of M. G. de Mortillet, was *le Moustier*, situated on the left bank of the Vézère (Dordogne). During its habitation by man the climate was cold and damp, and among the contemporary

personal protection of some kind. Our knowledge of their physique and general appearance is, as already mentioned, mainly derived from a comparison of a few of their fossil skeletons with those of modern civilised races. On this phase of the subject we have a considerable amount of evidence to show that since man parted company with the lower animals, there has been a gradual expansion of the cranium, corresponding to an enlargement of certain portions of the organ of thought. All such materials have, however, to be carefully sifted and scrutinised before being admitted as valid assets in a scientific inquiry; and even then, this kind of evidence seldom amounts to more than probability without being corroborated by other discoveries. The subject has grown so much of late that it was impossible in the limits at my disposal to do more than give a few pertinent examples. The race represented by the skulls of Neanderthal and Spy was long anterior to the time of the Palæolithic hunters of the reindeer period, who so greatly distinguished themselves as artists; and as to the Java skull and femur, they are probably the oldest osseous relics of man yet known. The human remains found in the rock-shelter of Cro-Magnon have been for a long time regarded as belonging to, and typical of, the latest Palæolithic people; but as they were merely lying over the culture-débris, they are regarded by some archaeologists as burials of a more recent date. The fact that the last molars were smaller than the others gives additional support to this view. It does not, however, appear to me that this point is of much consequence, as the amount of superincumbent talus under which the skeletons lay shows that they could not be later than the transition period. Moreover, there are other human remains with regard to which no such doubts have been raised, as, for example, the well-known skulls of Chancelade and Laugerie Basse, both found in the Dordogne district, which show equally advanced cranial characters.

The recent discovery of two skeletons, which Dr Verneau, of Paris, describes as belonging to a new race intermediate between the Neanderthaloid and Cro-Magnon races, marks an important addition to fossil craniology. From the preliminary facts already published, and from what Dr Verneau has told me, anthropologists may look forward with high expectation to the full report of these



to the progress of civilisation. The beneficial effect of this uncongenial environment on these early pioneers of humanity was to stimulate their natural capabilities of improvement—for the adage that necessity is the mother of invention was as applicable then as now. Entering Europe as naked, houseless nomads, living on wild fruits and the smaller fauna of a sub-tropical climate, they were ultimately forced by the severity of the climate to take refuge in caves and rock-shelters and to cover their bodies with skins. The natural food productions of a warm climate gradually disappeared, until finally there was little left but fierce animals, such as the mammoth, reindeer, chamois, horse, bison, etc., which came from northern regions into Central Europe. To procure the necessary food and clothing in these circumstances greatly taxed the skill and resources of the inhabitants. But this difficulty they ultimately solved by the manufacture of special weapons of the chase, with which they successfully attacked the larger wild animals which then occupied the country. The *coup de poing*, which for a long time served all the purposes of primitive life, gradually gave place to spear- and lance-heads fixed on long handles, together with a great variety of minor weapons and tools made of stone, bone, horn and wood. When the Palæolithic people finally emerged from this singular contest with the forces of nature, they were physically and mentally better than ever equipped for the exigencies of life. A greater power of physical endurance, improved reasoning faculties, an assortment of tools adapted for all kinds of mechanical work, and some experience of the advantage of housing and clothing, may be mentioned among the trophies which they carried away from that long and uphill struggle.

The civilisation thus developed represents the outcome of a system of human economy founded on the free play of natural laws, and little affected by the principles of religion or ethics—subjects which were as yet in their embryonic stage. The mysteries of the supernatural had not then been formulated into the concrete ideas of gods or demons. The notions of good and evil, right and wrong, were still dominated by the cosmic law that might is right. Neither gloomy forebodings nor qualms of conscience had much influence on the actions

of so many art specimens is of considerable importance among the more notable facts disclosed by these anthropological researches, as it proves that the origin of the artistic faculty was independent of, and prior to, the evolution of religion, ethics, politics, commerce, and other elements of which our modern civilisation is built up.

The other characteristic feature in the lives of these people was, that they lived exclusively on the produce of the chase, for, without agricultural and pastoral avocations, what else could they do but organise daily hunting or fishing expeditions? To capture the big game of the district was a formidable task, requiring not only great strength and agility of person and limb, but also strong and well-made weapons. During the later stages of the Palæolithic civilisation their principal prey consisted of reindeer and horses, both of which animals then roamed in large herds throughout Western Europe, thus rendering themselves more liable to be ambushed, trapped or speared by their wily enemies. It is not likely that they would take the initiative in attacking the hyæna, lion, or cave-bear, except in self-defence. That, however, these formidable creatures were occasionally captured by them is suggested by the fact that their canine teeth were highly prized as personal ornaments, or as a memento of their prowess in the chase. The weapons used by these hunters were harpoons, generally made of reindeer-horn, spear- and lance-heads of flint, and short daggers of bone or horn. Before these weapons were invented it is difficult to imagine that any member of the genus *Homo* would have the courage to attack such a formidable animal as the mammoth armed only with a *coup de poing*, but yet there are facts which suggest that such was the case.

When the physical conditions which called these accomplishments into existence passed away, and the peculiar fauna of the glacial period disappeared from the lowlands of Central Europe—some by extinction, and others by emigration to more northern regions or to the elevated mountains in the neighbourhood—we find the inhabitants of these old hunting grounds in possession of new and altogether different sources of food. Finding the former supplies becoming so limited and precarious that it was

necessary to hunt the animals in primeval forests. Skin-coats, dug-outs and stone weapons are now lineally represented by woven fabrics, Atlantic liners and Long Toms.

Were it possible for one of our Palæolithic ancestors to sit in judgment on the comparative merits of the two civilisations, I fancy his verdict would be something like the following: "You have utilised the forces of nature to a marvellous extent, and thereby greatly increased the means of subsistence to your fellow-creatures; but, at the same time, you have facilitated the physical degeneracy of your race by multiplying the sources of human disease and misery. The invention of money has facilitated the accumulation and transmission of riches to a few; but it has impoverished the many, and supplied incentives to fraud, theft, and all manner of crime. Patriarchal establishments have given place to social organisations, governed by laws founded on moral sentiments and ethics; but their by-products are extreme luxury and extreme poverty. Hence, to support the weak and the unfortunate is no longer a matter of charity, but a legal and moral obligation. Notwithstanding the size of your asylums, hospitals and almshouses, they are always full and always on the increase. Your legislators are selected by the voice of the majority: what if that majority be steeped in superstition, prejudice and ignorance? You have formulated various systems of religion, but whether founded on the principles of fetichism, polytheism or monotheism, they are still more or less permeated with contradictory or controvèrted creeds and dogmas. Natural sport, as practised with weapons of modern precision, can only be characterised as legalised killing of helpless creatures. To shoot pigeons suddenly liberated from a box at a measured distance, or overfed pheasants, even after they have managed to take wing, or semi-domesticated deer, especially when driven to the muzzle of a rifle—all, of course, within sight of a luncheon basket—is a poor substitute for the excitement and field incidents of the chase in Palæolithic times. With no better weapons than a spear, or lance tipped with a pointed flint, and a small dagger of bone or horn, we had, not infrequently, to encounter in mortal combat the mammoth, rhinoceros, cave-bear, or some other fierce and hungry animal, which, like ourselves, was prowling in quest of a morning meal. Such

in London and Paris. Nos. 1-7, 9-11, 18 and 19 represent saws, borers, scrapers, etc. from the later stations. Nos. 12 and 16 are illustrations of the laurel-leaf-shaped lance-heads commonly described as belonging to the *Solutréen* period. The former was found at Laugerie Basse (Col. Massénat-Girod), and the latter (made of agate) in the Grotte de l'Église (Dordogne). Nos. 8, 15, 17 and 21 are specimens of the earlier implements from Le Moustier, and are all trimmed flakes, with the exception of 17, which is a small *coup de poing*. No. 18 represents a core from Les Eyzies, showing on the left a small portion of the original surface of the flint, and No. 20 a well-made flake from La Madelaine. A small mortar made out of a waterworn pebble from Les Eyzies is shown under figure 14; others like it have been recorded from La Madelaine, Laugerie Basse, Bruniquel, and probably elsewhere.

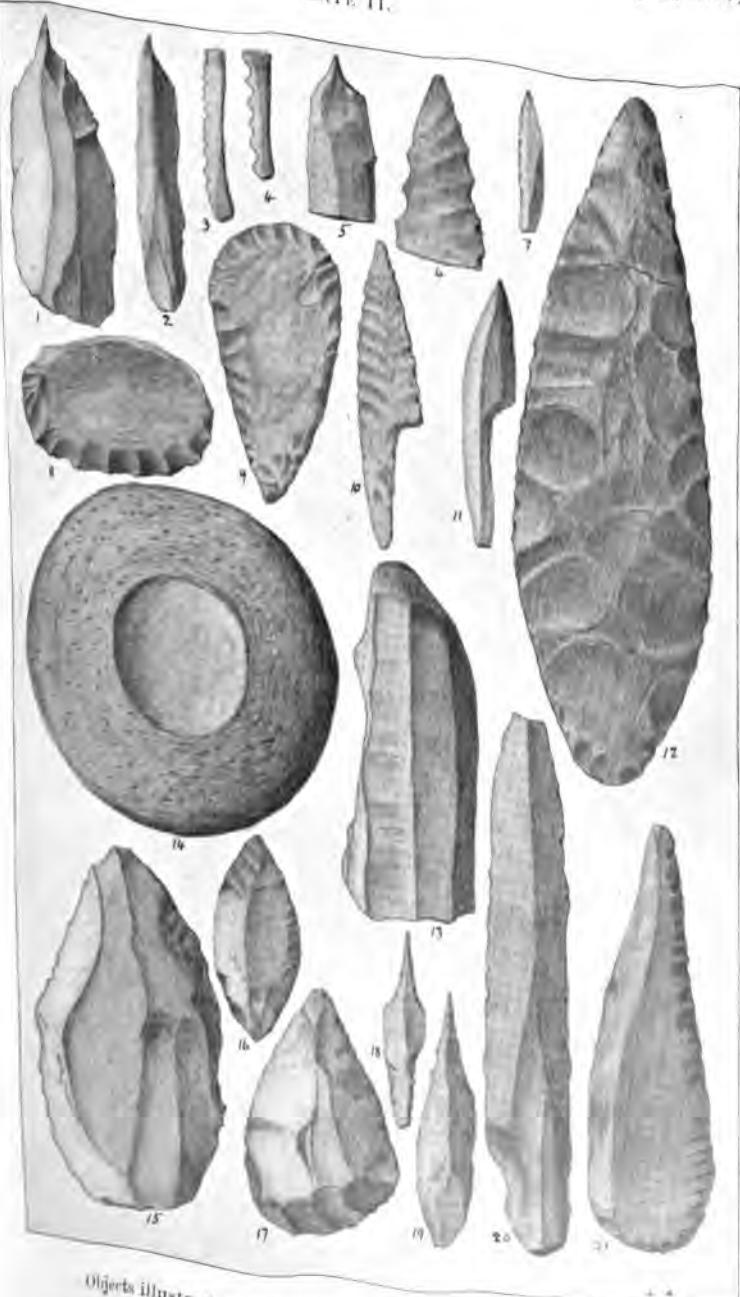
III. Weapons and ornaments made of bone, teeth, deer-horn, ivory and shells. Nos. 1-14, 15, 17-19 (ivory), 20, 25 (ox), 26 (fox), 27 and 28 are from La Madelaine (Col. L. and C.). Nos. 5-14 are from Laugerie Basse (Col. Massénat-Girod). Nos. 24 and 29, representing a supposed whistle and a sculptured dagger, are from Laugerie Basse (Col. L. and C.). No. 16 is a thin plaque carved of bone, probably an ornamental pendant, found at Bruniquel (British Museum). Nos. 21-23 are from Kent's Cavern. The precise use of the pointed objects figured under Nos. 12-14, 28 and 30 is not known, but it is probable that they were the tips of small lances propelled by means of such an implement as is figured under No. 8, Plate IV. The small harpoon (No. 27) might have been used as an arrow-point, but we have no evidence that bows and arrows were then in use.

IV. On this Plate there is a collection of objects from various stations illustrating the art of the Palæolithic people. No. 1 shows a portion of reindeer-horn with a rude representation of a prone man, apparently in the act of throwing a spear at a male auroch. The hands are imperfectly represented, the body is covered with hair, and a cord, possibly attached to the head of a harpoon, falls behind the legs. This specimen was found at Laugerie Basse (Col. Massénat-Girod). Nos. 2 and 14 represent portions of darts with badly-executed human hands, showing only four fingers. Nos. 3, 4 and 5 are from La Madelaine (Col. L. and C.). One (3) represents a piece of stag's horn (*bâton de commandement*), having a stag with complex antlers incised on it. Another (4) is a plate of the canon bone of a reindeer with incised figures of bovine animals. The third represents a truncated dart ornamented with flowers, and what looks like the outstretched skin of a fox. No. 6 is from Les Eyzies, and shows a ruminant having a spear entering its breast (*ibid.*). A portion of a bevelled dart-head from Laugerie Basse, with a sequence of half-fledged birds, is shown by No. 7 (*ibid.*). No. 8 represents a dart-propeller from Laugerie Basse, ornamented with a horse's head and an elongated forepart of a deer (*ibid.*). Nos. 9, 10 and 15 are also from Laugerie Basse (Col. Massénat-Girod), and represent the well-extended antlers of a reindeer (9), an otter eating a salmon (10), and a hare (15), sculptured in ivory. No. 11, unmistakably showing the hind portion of a pig, is from the Kesslerloch, Switzerland (after Conrad Merk). On the canine of a bear (No. 12) from Duruthy Cave a seal is engraved (*Reliquiae Aquitanicae*, p. 223). The palm of the brow antler of a reindeer is incised with the figure of some kind of horned animal (No. 13), probably intended for an ibex.



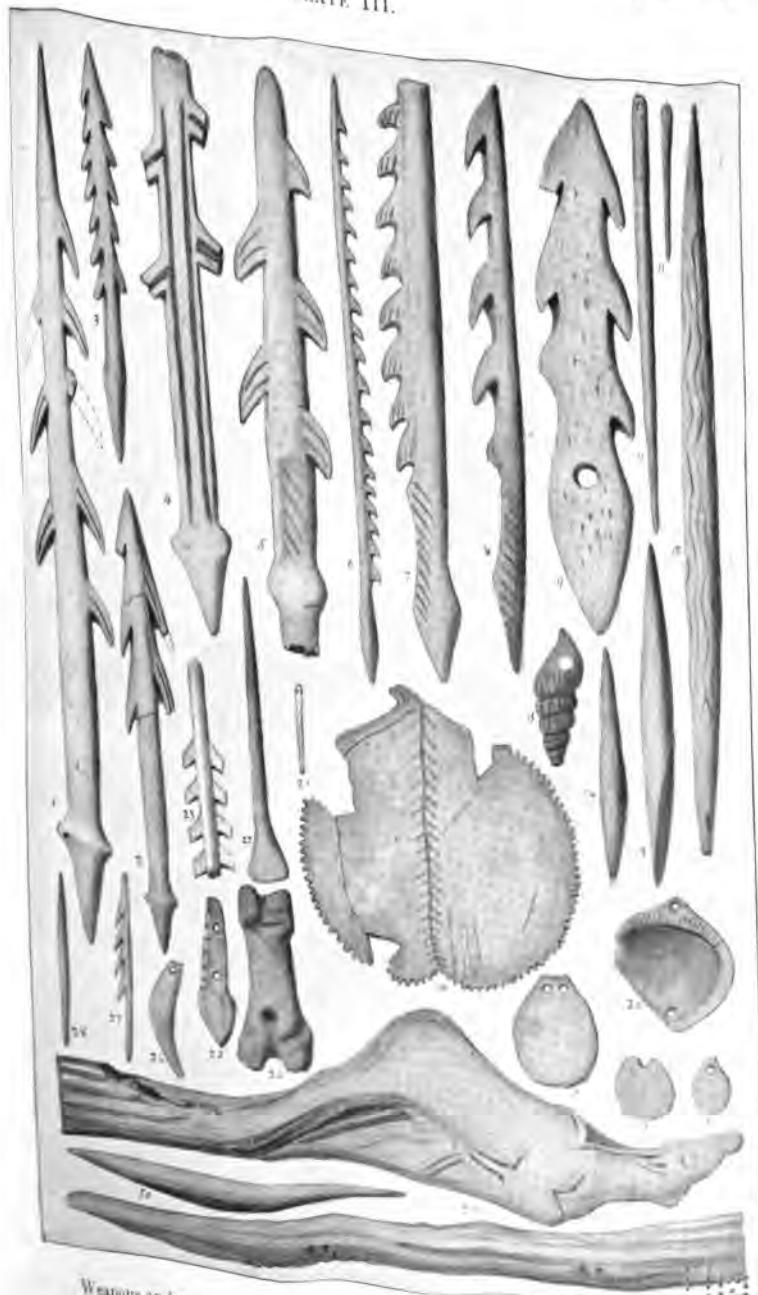


Flint implement—^{*} coup de poing²—from river-drift gravels (1).
Dr. MYSRO.

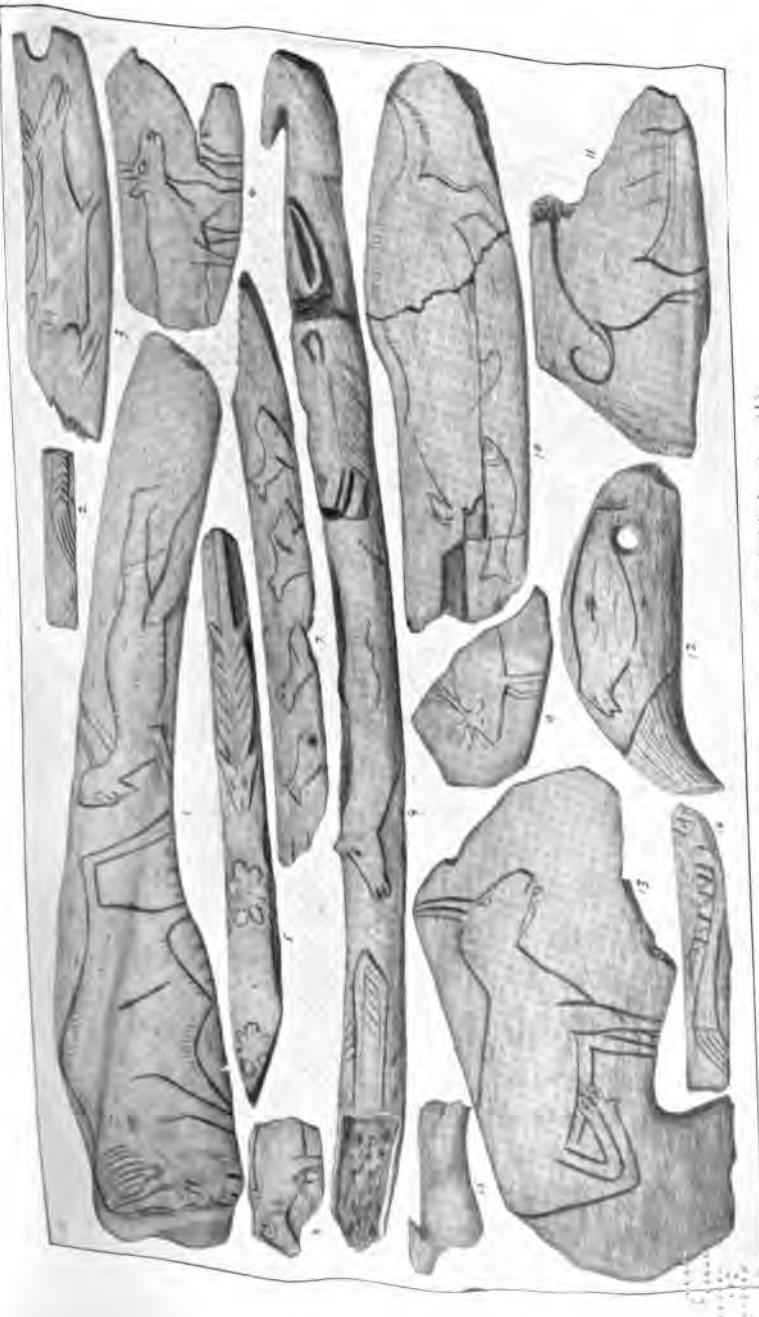


Objects illustrating the industry among the Cave men of France (1).

Dr. MUSEO.



Weapons and ornaments made of boar's teeth, deer-horn, ivory and shells (1).
THE MUSEUM.



Proc. Roy. Socy. of Edin.]

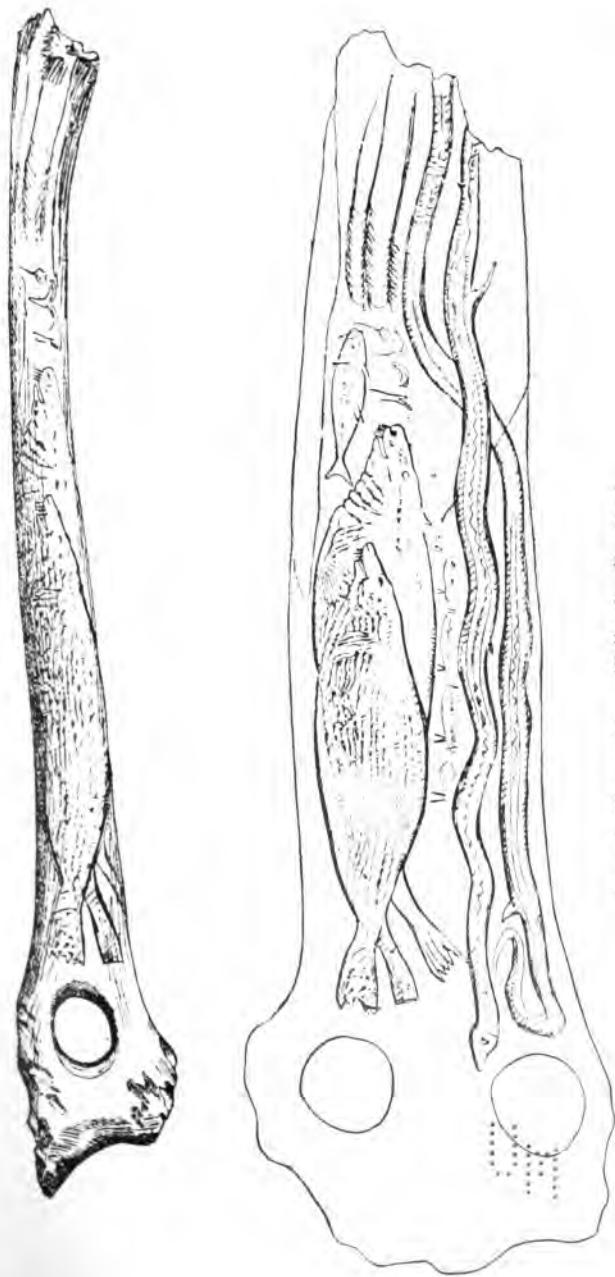
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PLATE V.



Mammoth engraved on a piece of ivory, La Madeleine (J). (E. Lartet.)

Dr. MUNRO.



Ritton de Montgoulier. (Collection Pognon.)

in Musée.



FIG. 1.—Human figure, horse and serpent ($\frac{1}{2}$).



FIG. 2.—Horses in sequence ($\frac{1}{2}$).
Two *bâtons de commandement* from La Madeleine;
Dh. Mésnil.

PLATE VIII.

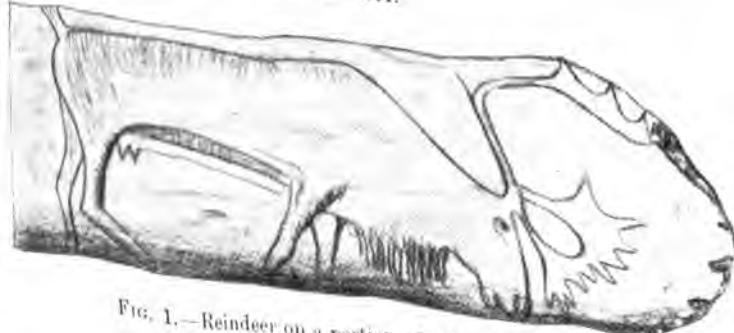


FIG. 1.—Reindeer on a portion of reindeer-horn (1).

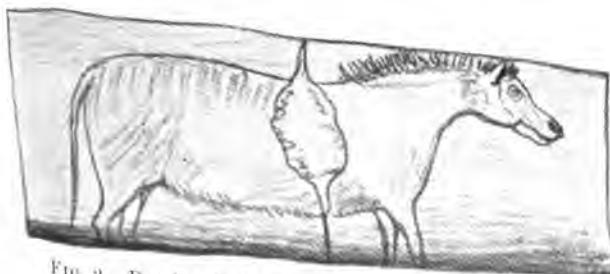


FIG. 2.—Drawing of a horse on portion of reindeer-horn (1).



FIGS. 3, 4, 5.—A perforated shell and hanging ornaments made of coal (1).

Engraved figures of animals and ornaments from the Kesslerloch Cave, near Schaffhausen. (After Conrad Merk.)

DR. MUNRO.



Fig. 1.—Handle of a dagger sculptured into the form of a reindeer.
Rock-shelter of Bruniquel ($\frac{2}{3}$).



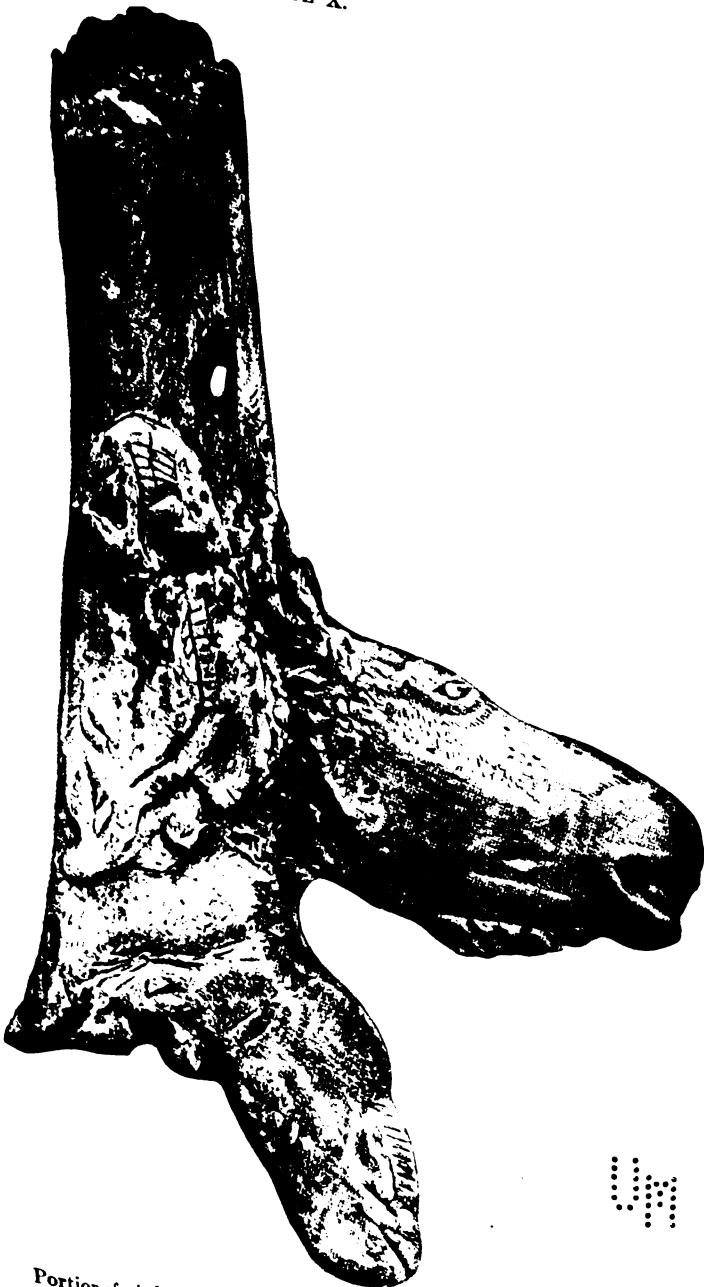
Fig. 2.—Mammoth sculptured in reindeer-horn. Rock-shelter of Bruniquel ($\frac{2}{3}$).



Fig. 3.—Unknown animal sculptured in reindeer-horn. Laugerie Basse (1/2).
Animals sculptured in ivory and horn.

On Musée.

PLATE X.



Up

Portion of reindeer-horn from Mas d'Azil, sculptured into two horse-heads
(Col. Piette). After E. Cartailhac—*La France Prehistorique*.

Dr. MUNRO.

PLATE XI



FIG. 1.—Bison painted in oehre.



FIG. 2.—Reindeer partly painted and partly incised.
Specimens of painted animals from the Cave of Font-de-Gaume, after MM.
Capitan and Breuil.

DR. MUNRO.





LEVI, Leon.—On Deep-water Two-dimensional Waves produced by any given Initiating Disturbance.

Proc. Roy. Soc. Edin., vol. xxv., 1904, pp. 185-196.

WAVE, Deep-water Two-dimensional, produced by any given Initiating Disturbance.

LOD KELVIN.

Proc. Roy. Soc. Edin., vol. xxv., 1904, pp. 185-196.

The Theory of *Continuants* in the Historical Order of its Development up to 1870. By Thomas Muir, LL.D.

(MS. received October 5, 1903. Read November 2, 1903.)

The more or less disguised use of continued fractions has been traced back to the publication of Bombelli's *Algebra* in 1572, eighty-four years, that is to say, before the publication of Wallis' *Arithmetica Infinitorum*, in which Brouncker's discovery was announced and the fractions explicitly expressed.* The study of the numerators and denominators of the convergents viewed as functions of the partial denominators was first seriously undertaken by Euler in his *Specimen Algorithmi Singularis* of the year 1764, in which denoting by

$$(a), \frac{(a, b)}{(b)}, \frac{(a, b, c)}{(b, c)}, \dots$$

the convergents to

$$a + \frac{1}{b} + \frac{1}{c} + \dots$$

he established a long series of identities, such as

$$(a, b, c, d, \dots) = a(b, c, d, \dots) + (c, d, \dots)$$

$$(a, b, c, \dots, l) = (l, \dots, c, b, a),$$

$$(a, b)(b, c) - (b)(a, b, c) = 1,$$

$$(a, b, c)(d, e, f) - (a, b, c, d, e, f) = -(a, b)(e, f),$$

$$\dots \dots \dots \dots \dots$$

The study was pursued by Hindenburg and his followers during the last twenty years of the eighteenth century, but not with any great profit; and, although in the first half of the nineteenth century considerable attention was given to the theory of continued fractions as a whole, little advance was made in elucidating

* For the early history see Favaro's *Notizie storiche sulle frazioni continue dal secolo decimoterzo al decimosettimo* published in vol. vii. of Boncompagni's *Bollettino*: and as regards Bombelli see a paper by G. Wertheim in the *Abhandl. zur Gesch. d. Math.*, viii. pp. 147-160.

(the mode of formation of which is self-apparent), these successive coaxal determinants will be

$$1, A_1, \begin{vmatrix} A_1 & 1 \\ 1 & A_2 \end{vmatrix}, \begin{vmatrix} A_1 & 1 & 0 \\ 1 & A_2 & 1 \\ 0 & 1 & A_3 \end{vmatrix}, \begin{vmatrix} A_1 & 1 & 0 & 0 \\ 1 & A_2 & 1 & 0 \\ 0 & 1 & A_3 & 1 \\ 0 & 0 & 1 & A_4 \end{vmatrix}, \text{ etc.}$$

i.e.

$$\begin{aligned} 1, A_1, & A_1 A_2 - 1, A_1 A_2 A_3 - A_1 - A_3, \\ A_1 A_2 A_3 A_4 - A_1 A_2 - A_1 A_4 - A_3 A_4 + 1, \\ A_1 A_2 A_3 A_4 A_5 - A_1 A_2 A_5 - A_1 A_4 A_5 - A_2 A_4 A_5 - A_1 A_2 A_3 \\ & + A_5 + A_3 + A_1. \end{aligned}$$

It is proper to introduce the unit because it is, in fact, the value of a determinant of zero places, as I have observed elsewhere."

After using this as an aid to prove his proposition regarding Sturm's theorem, he returns to his new determinant in the following words:—

"I may conclude with noticing that the determinative [determinantal?] form of exhibiting the successive convergents to an improper continued fraction affords an instantaneous demonstration of the equation which connects any two consecutive such convergents as

$$\frac{N_{i-1}}{D_{i-1}} \text{ and } \frac{N_i}{D_i} \text{ viz. } N_i D_{i-1} - N_{i-1} D_i = 1.$$

For if we construct the matrix which for greater simplicity I limit to five lines and columns,

A	1	0	0	0
1	B	1	0	0
0	1	C	1	0
0	0	1	D	1
0	0	0	1	E

and represent umbrally as

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{matrix};$$

and we shall thus, by the same process as has been applied to improper continued fractions, obtain

$$N_{i+1}D_i - N_iD_{i+1} = (\sqrt{-1})^i \times (\sqrt{-1})^i \\ = (-1)^i.$$

This would seem to imply that as yet Sylvester had not observed that an alternative mode of representation was obtainable by merely changing the sign of the units on one side of the diagonal.

The footnote contains two additional observations, the first being to the effect that the new mode of representation

"gives an immediate and visible proof of the simple and elegant rule for forming any such numerators or denominators by means of the principal terms [term $\overline{1}$] in each; the rule, I mean, according to which the i^{th} denominator may be formed from

$$q_1 q_2 q_3 q_4 \dots q_i$$

(q_1, q_2, \dots, q_i being the successive quotients) and the i^{th} numerator from

$$q_2 q_3 q_4 \dots q_i$$

by leaving out from the above products respectively any pair or any number of pairs of consecutive quotients as $q_p q_{p+1}$.

For instance, from $q_1 q_2 q_3 q_4 q_5$ by leaving out $q_1 q_2, q_2 q_3, q_3 q_4$ and $q_4 q_5$ we obtain

$$q_3 q_4 q_5 + q_1 q_4 q_5 + q_1 q_2 q_5 + q_1 q_2 q_3 :$$

and by leaving out $q_1 q_2 \cdot q_3 q_4, q_1 q_2 \cdot q_4 q_5, q_2 q_3 \cdot q_4 q_5$ we obtain

$$q_5 + q_3 + q_1;$$

so that the total denominator becomes

$$q_1 q_2 q_3 q_4 q_5 + q_3 q_4 q_5 + q_1 q_4 q_5 + q_1 q_2 q_5 + q_1 q_2 q_3 + q_5 + q_3 + q_1;$$

and in like manner the numerator of the same convergent is

$$\text{i.e. } q_2 q_3 q_4 q_5 \left\{ 1 + \frac{1}{q_2 q_3} + \frac{1}{q_3 q_4} + \frac{1}{q_4 q_5} + \frac{1}{q_2 q_3 q_4 q_5} \right\}$$

$$q_2 q_3 q_4 q_5 + q_4 q_5 + q_2 q_5 + q_2 q_3 + 1.$$

The "rule" here spoken of is that enunciated for the more general case of

$$a_1 + \frac{b_1}{a_2} + \frac{b_2}{a_3} +$$

in Stern's *Theorie der Kettenbrüche* (1858), given up to the consideration of the first four terms (pp. 4-7).

The other observation is that every progression can be reduced to the equation

$$u_n = a_n u$$

which may be represented as a system of determinants. If the progression is continued indefinitely, it is represented by a matrix

indefinitely continuing the series
1, A, AA' -

This exhausts the part of the results announced by Stern, and fairly entitles him to entitle the entire article "the introduction of the theory of continued fractions upon the future treatment of numbers."

SPOT

[Elementary theory of continued fractions, rewritten and published in the *Journ. für Math.*, 11. (1858).]

Save the utilization of the convergent of the continued fraction

* This is the author's part of the volume, however.

is the differential-quotient of the numerator, Spottiswoode did nothing but report the fundamental result reached by Sylvester. The full passage (p. 374) is as follows:—

"The improper continued fraction

$$\frac{1}{A - \frac{1}{B - \frac{1}{C - \dots}}} = \frac{d}{dA} \log_e \nabla$$

where

$$\nabla = \begin{vmatrix} A & 1 & 0 & \dots & 0 & 0 \\ 1 & B & 1 & \dots & 0 & 0 \\ 0 & 1 & C & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M & 1 \\ 0 & 0 & 0 & \dots & 1 & N \end{vmatrix}$$

in which any number of rows may be taken at pleasure, and the formula will give the corresponding convergent fraction.

The same holds good for the continued fraction

$$\frac{1}{A + \frac{1}{B + \dots}}$$

if we write

$$\nabla = \begin{vmatrix} A & 1 & 0 & \dots \\ 1 & B & 1 & \dots \\ 0 & 1 & C & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix},$$

SYLVESTER, J. J. (1853, Sept.).

[On a fundamental rule in the algorithm of continued fractions.
Philos. Mag. (4), vi. pp. 297-299.]

Without any reference to his previous paper on the subject Sylvester here announces that if

$$(a_1, a_2, \dots, a_i)$$

be the denominator of the i^{th} convergent to

$$\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

then

$$(a_1, \dots, a_m, a_{m+1}, \dots, a_{m+n}) = (a_1, \dots, a_m)(a, + (a_1, \dots, a_{m-1})$$

—a possibly new result which he considers theorem in the theory of continued fractions. an immediate consequence of the fact that (can be expressed as a determinant, all that is further the application of the “well-known simple relation of determinants.” Thus, e.g., the deter-

$$\begin{array}{ccccc} a & 1 & & & \\ -1 & b & 1 & & \\ -1 & c & 1 & & \\ -1 & d & 1 & & \\ -1 & e & & & \\ -1 & & & & \end{array}$$

is obviously decomposable into

$$\begin{array}{cc} a & 1 \\ -1 & b \\ -1 & c \end{array} \times \begin{array}{cc} d & 1 \\ -1 & e \\ -1 & f \end{array}$$

or into

$$\begin{array}{cc} a & 1 \\ -1 & b \\ -1 & c \end{array} \times \begin{array}{cc} e & 1 \\ -1 & d \\ -1 & e \\ -1 & \end{array}$$

or into

$$\begin{array}{cc} a & \\ -1 & \end{array} \times \begin{array}{cc} b & 1 \\ -1 & c \\ -1 & d \\ -1 & \end{array}$$

Following this is what is
 $(a_1, a_2, \dots, a_m) \cdot (a_2, a_3, \dots, a_n)$

its connection with the e
 ts being illustrated

The next "corollary," viz.,

$$\begin{aligned} & (a_1, \dots, a_p, a_{p+1}, \dots, a_{p+f})(a_1, \dots, a_p, a_{p+1}, \dots, a_{p+h}) \\ & - (a_1, \dots, a_p, a_{p+1}, \dots, a_{p+g})(a_1, \dots, a_p, a_{p+1}, \dots, a_{p+h}) \\ & = (-)^g \{ (a_{p+1}, \dots, a_{p+f})(a_{p+1}, \dots, a_{p+h}) - (a_{p+1}, \dots, a_{p+g})(a_{p+1}, \dots, a_{p+h}) \} \end{aligned}$$

is clearly incorrect, it being impossible for the value of the left-hand side to be independent of the elements a_1, a_2, \dots, a_p . Further, as the author gives no accompanying word of comment, the difficulty of suggesting the true theorem is increased. A "sub-corollary" is appended dealing with the case where all the a 's are equal, and leading up, not without some misprints or inaccuracies, to a theorem of Euler's quoted from the *Nouvelles Annales de Math.*, v. (Sept. 1851) pp. 357-358, to the effect that if $T_{n+1} = aT_n - bT_{n-1}$ be the generating equation of a recurrent series, then

$$\frac{T_{n+1}^2 - aT_n T_{n+1} + bT_n^2}{b^n}$$

is a constant with respect to n . Of course the more natural form of this expression is

$$\frac{T_{n+1}^2 - T_n T_{n+2}}{b^n},$$

the numerator of which being

$$\begin{vmatrix} T_{n+1} & T_{n+2} \\ T_n & T_{n+1} \end{vmatrix}$$

is successively transformable by means of the recursion-formula into

$$b \begin{vmatrix} T_n & T_{n+1} \\ T_{n-1} & T_n \end{vmatrix}, \quad b^2 \begin{vmatrix} T_{n-1} & T_n \\ T_{n-2} & T_{n-1} \end{vmatrix}, \quad b^3 \begin{vmatrix} T_{n-2} & T_{n-1} \\ T_{n-3} & T_{n-2} \end{vmatrix}, \dots$$

so that the constant in question is

$$\begin{vmatrix} T_1 & T_2 \\ T_0 & T_1 \end{vmatrix}.$$

This, however, Sylvester does not show.*

* An interesting extension of this is given by Brioschi in the *Nouvelles Annales de Math.*, xiv. (Jan. 1854) p. 20.

Finally, and to more purpos
(a_1, a_2, \dots, a_i) to the readi

$$\begin{array}{cc} m_1 & l_1 \\ n_1 & m_2 \\ & \vdots \\ n_2 & r \end{array}$$

the corresponding fundamental

$$\begin{pmatrix} l_1, \dots, l_{i+j} \\ m_1, m_2, \dots, m_{i+j+1} \\ n_1, \dots, n_{i+j} \end{pmatrix} = \begin{pmatrix} l_1, \dots, \\ m_1, m_2, \dots \\ n_1 \end{pmatrix} - l_i n_i \begin{pmatrix} l_1, \dots, \\ m_1, m_2, \dots \\ n_1 \end{pmatrix}.$$

SYLVESTER, J. J

[On a theory of the syzyget
functions, comprising
Sturm's functions, and
common measure. *Ph*
pp. 407-548.]

Although this lengthy mem
“16th June 1853,” it is the eq
later while passing through th
present connection. In the fi
nominator of the fraction

$$\frac{1}{q_1} - \frac{1}{q_2}$$

is denoted by $[q_1, q_2, \dots, q_n]$
at the later portion of

"supplement" that anything apparently new in substance is met with. There in § a (p. 497) the following lemma occurs:—

"The roots of the cumulant $[q_1, q_2, \dots, q_i]$ in which each element is a linear function of x , and wherein the coefficient of x for each element has the like sign, are all real: and between every two of such roots is contained a root of the cumulant $[q_1, q_2, \dots, q_{i-1}]$ and *ex converso* a root of the cumulant $[q_2, q_3, \dots, q_i]$: and (as an evident corollary) for all values of ζ and ζ' intermediate between 1 and i the greatest root of $[q_1, q_2, \dots, q_i]$ will be greater, and the least root of the same will be less than the greatest and least roots respectively of $[q_p, q_{p+1}, \dots, q_{p'-1}, q_p]$."

Even this, however, may be placed under the well-known theorem regarding the roots of the equation

$$\left| \begin{array}{cccc} a_{11} - x & a_{12} & a_{13} & \dots \\ a_{12} & a_{22} - x & a_{23} & \dots \\ a_{13} & a_{23} & a_{33} - x & \dots \\ \dots & \dots & \dots & \dots \end{array} \right| = 0$$

which had been enunciated by Cauchy in 1829.*

The next noteworthy result occupies § i. (p. 502). As a preparation for it the theorem

$$[a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n] = [a_1, a_2, \dots, a_m][b_1, b_2, \dots, b_n] - [a_1, a_2, \dots, a_{m-1}][b_2, b_3, \dots, b_n]$$

may be recalled, the group of elements on the left being now viewed as consisting of two sub-groups. This theorem Sylvester writes in the form

$$[\Omega_1 \Omega_2 \Omega_3] = [\Omega_1][\Omega_2] - [\Omega'_1][\Omega_2]$$

and he succeeds in including in it a general theorem, not explicitly formulated, in which the number of groups is i , the next two cases being

$$[\Omega_1 \Omega_2 \Omega_3] = [\Omega_1][\Omega_2][\Omega_3] - [\Omega'_1][\Omega_2][\Omega_3] - [\Omega_1][\Omega'_2][\Omega_3] + [\Omega'_1][\Omega'_2][\Omega_3],$$

* a. The theory of orthogonants . . . in *Proc. Roy. Soc. Edinburgh*, xxiv. p. 261.

and

$$[\Omega_1 \Omega_2 \Omega_3 \Omega_4] = [\Omega_1] [\Omega_2] [\Omega_3] [\Omega_4] \\ - [\Omega'_1] [\Omega_2] [\Omega_3] [\Omega_4] \\ + [\Omega'_1] [\Omega'_2] [\Omega_3] [\Omega_4] \\ - [\Omega'_1] [\Omega'_2] [\Omega'_3] [\Omega'_4]$$

The general theorem is de
[$\Omega_1 \Omega_2 \dots \Omega_i$] in terms of

$$[\Omega_1], [\Omega_2], \\ [\Omega'_1], [\Omega'_2], \\ [\Omega'_2], \\ [\Omega'_2],$$

that is to say, in terms of
Ω's except the last, all the
the "doubly-apocopated" Ω's
is pointed out that the nu
pansion is 2^{i-1} "separable in
groups containing respectively
 $i-1, 1$ products." Further,
above groups forming a pro
between contiguous factors,
have an accent on the right to
and if it have one on the left
accent on the right, and the
increasing in each group from

In a footnote the case when
where, therefore, each singly
doubly-accented Ω vanishes,

"rule"

$$[a_1, a_2, \dots, a_i] = a_1 a_2 \dots a_i \\ + \dots$$

SYLVESTER, J. J. (1854, August).

[Théorème sur les déterminants de M. Sylvester. *Nouv. Annales de Math.*, xiii. p. 305.]

This communication in its entirety is as follows:—

“Soient les déterminants

$$\begin{array}{ccccccccc}
 \lambda, & \lambda & 1 & \lambda & 1 & 0 & \lambda & 1 & 0 \quad 0 \\
 & 1 & \lambda, & 2 & \lambda & 2 & 3 & \lambda & 2 \quad 0 \\
 & & & 0 & 1 & \lambda, & 0 & 2 & \lambda \quad 3 \\
 & & & & & & 0 & 0 & 1 \quad \lambda, \\
 & & \lambda & 1 & 0 & 0 & 0 & & \\
 & & 4 & \lambda & 2 & 0 & 0 & & \\
 & & 0 & 3 & \lambda & 3 & 0 & & \\
 & & 0 & 0 & 2 & \lambda & 4 & & \\
 & & 0 & 0 & 0 & 1 & \lambda, & \dots \dots
 \end{array}$$

la loi de formation est évidente; effectuant, on trouve

$$\begin{aligned}
 \lambda, \lambda^2 - 1, \lambda(\lambda^2 - 2^2), (\lambda^2 - 1^2)(\lambda^2 - 3^2), \lambda(\lambda^2 - 2^2)(\lambda^2 - 4^2), \\
 (\lambda^2 - 1^2)(\lambda^2 - 3^2)(\lambda^2 - 5^2), \lambda(\lambda^2 - 2^2)(\lambda^2 - 4^2)(\lambda^2 - 6^2),
 \end{aligned}$$

et ainsi de suite.”

That Sylvester was the author of the implied theorem may be considered proved by an entry in the index of the volume (v. p. 478), and by a statement of Cayley's in the *Quarterly Journal of Mathematics*, ii. p. 163. Probably the title of the communication was prefixed by the editors, who, knowing of Sylvester's papers in the *Philosophical Magazine*, felt themselves justified in applying the name “Sylvester's determinants.”

SCHLÄFLI, L. (Nov. 1855).

[Réduction d'une intégrale multiple qui comprend l'arc de cercle et l'aire du triangle sphérique comme cas particuliers. *Journ. de Liouville*, xx. pp. 359-394.]

Here there appears the equation

$$\frac{\Delta(a, \beta, \dots, \zeta, \eta)}{\Delta(\beta, \dots, \zeta, \eta)} = 1 - \frac{\cos^2 a}{1} - \frac{\cos^2 \beta}{1} - \dots - \frac{\cos^2 \zeta}{1 - \cos^2 \eta}$$

where, in view of the c
year 1858), it would see

$$\begin{array}{ccc} 1 & \cos \alpha & \\ -\cos \alpha & 1 & \\ & -\cos \alpha & \end{array}$$

No properties, however

[Determinanternes
convergerend
Forhandlinger]

Ramus' introduction
of determinants
his mode of stating
Deformatione . . .
his set of equations:

$$\begin{array}{l} a_0^0 y_0 + \\ a_0^1 y_0 + \\ \cdot \cdot \cdot \\ a_0^n y_0 + \end{array}$$

and puts the solution
 $R_n y_r = A_r^0$

where

$$\begin{aligned} R_n &= \sum \pm a_0 \\ A_r^i &= \sum \pm a \\ A_r^i &= - \sum \pm \end{aligned}$$

* It is in this
peculiarity consisting
column in which

He then recalls the further fact that if $y_0, y_1, y_2, \dots, y_n$ be the numerators of the convergents of the continued fraction

$$a_0 + \frac{b_1}{a_1 + a_2 + \dots + \frac{b_n}{a_n}}$$

there exists the set of equations

$$\left. \begin{aligned} y_0 &= a_0 \\ -a_1 y_0 + y_1 &= b_1 \\ -b_2 y_0 - a_2 y_1 + y_2 &= 0 \\ -b_3 y_1 - a_3 y_2 + y_3 &= 0 \\ \dots & \dots \\ -b_n y_{n-2} - a_n y_{n-1} + y_n &= 0 \end{aligned} \right\}$$

and he thereupon draws the natural conclusion that the previous result can be applied to the determination of $y_0, y_1, y_2, \dots, y_n$.

Making the necessary substitution for the u 's and for R_n he of course obtains

$$y_n = a_0 A_n^0 + b_1 A_n^1,$$

A_n^0, A_n^1 being now determinants which for want of Cayley's notation he cannot accurately specify, but which he persists in writing in the form

$$- \sum \pm a_0^n a_1^1 a_2^2 \dots a_{n-1}^{n-1}, \quad - \sum \pm a_0^0 a_1^n a_2^2 \dots a_{n-1}^{n-1}.$$

From this result he calculates in succession the values of y_1, y_2, y_3, y_4 ; but it will readily be understood that the process is neither elegant nor short.

In the remainder of the paper (§§ 4-9) no further use of the properties of determinants is made, the contents of the last ten pages being such as might appear in any ordinary exposition of continued fractions. First there is established the old "rule" for writing out the value of y_n , above referred to as being given by Stern. This is followed by the results

factor $(-1)^{i+\kappa}$, he takes the further step of moving the row with the index κ over $\kappa-i+1$ rows, thus arriving at

$$A_\kappa^i = - \sum \pm a_0^0 a_1^1 \dots a_{i-1}^{i-1} a_{i+1}^{i+1} \dots a_{\kappa-1}^{\kappa-1} a_i^\kappa a_{\kappa+1}^{\kappa+1} \dots a_n^n.$$

Of course there is at the second step the option of moving the column with the index i over $\kappa-i+1$ columns, and this Ramus does.

$$a + \frac{b}{a + \frac{b}{a + \frac{b}{\ddots (n b's)}}} = \frac{1}{\sqrt{a^2 + b^2}}$$

which by putting $a=1$ =
number also obtained in t

$$\frac{1}{2^n} \left\{ C_{n+2,1} + \right.$$

Anything else is of sm:

CAYL
[On the determinatio
Quart. Journ. of
Papers, iii. pp. 15

The determinant in c
vester's of the year 185
being

$$\begin{vmatrix} \theta & 1 & & \cdot \\ x & \theta & & 2 \\ \cdot & x-1 & \theta & \\ \cdot & \cdot & x & \\ \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \end{vmatrix}$$

while the other is obt
noting his own form by
him, found

$$U_2 = (\theta^2 - 1) - (x)$$

$$U_3 = \theta(\theta^2 - 4) - :$$

$$U_4 = (\theta^2 - 1)(\theta^2 - 9)$$

so that, if he put H_n fo
when $x = n - 1$), he cou

$$U_2 = H_2 - ($$

$$U_3 = H_3 - :$$

$$U_4 = H_4 - :$$

.....

and thence, doubtless, divined the generalisation

$$U_n = H_n - B_{n,1} \cdot (x - n + 1) \cdot H_{n-2} + B_{n,2} \cdot (x - n + 1)(x - n + 3) \cdot H_{n-4} - \dots$$

where

$$H_n = (\theta + n - 1)(\theta + n - 3)(\theta + n - 5) \dots \text{to } n \text{ factors}$$

$$B_{n,s} = \frac{n(n-1)(n-2) \dots (n-2s+1)}{2 \cdot 1 \cdot 2 \cdot 3 \dots s}.$$

The establishment of the truth of this is all that the paper is occupied with, the procedure being to expand U_n in terms of the elements of its last row and their complementary minors, thus obtaining

$$U_n = \theta U_{n-1} - (n-1)(x - n + 2) U_{n-2}$$

and thence

$$U_n + \left\{ (n-1)(x - n + 2) + (n-2)(x - n + 3) - \theta^2 \right\} U_{n-2} + (n-2)(n-3)(x - n + 3)(x - n + 4) U_{n-4} = 0,$$

and showing that the above conjectural expression for U_n satisfies the latter equation. The process of verification is troublesome, and was not viewed with satisfaction by Cayley himself.

As a preliminary the coefficients of the H 's in the value of U_n are for shortness' sake denoted by $A_{n,0}, -A_{n,1}, \dots$, and for the same and an additional reason the coefficient of U_{n-2} in the difference-equation is denoted by

$$M_{n,s} = \left\{ \theta^2 - (n-2s-1)^2 \right\},$$

which is equivalent to putting

$$M_{n,s} \equiv (n-1)(x - n + 2) + (n-2)(x - n + 3) - (n-2s-1)^2.$$

The operation to be performed being thus the substitution of

$$A_{n,0}H_n - A_{n,1}H_{n-2} + \dots + (-)^s A_{n,s}H_{n-2s} + \dots$$

for U_n in the expression

$$U_n + \left[M_{n,s} - \left\{ \theta^2 - (n-2s-1)^2 \right\} \right] U_{n-2} + (n-2)(n-3)(x - n + 3)(x - n + 4) U_{n-4},$$

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it is readily seen that the like

$$A_{n,s}H_{n-2s} + \left[M_n + (n-2)(n-1)$$

Now if we bear in mind

$$\left\{ \theta^2 - ($$

the second of the three

$$= M_n$$

or, if we write $s = 1$ for

$$= - M$$

$$= - H$$

and the third, by writing

$$= (n-2)(n-1)$$

Consequently the sum

$$A_{n,s} - (M_{n,s-1}A_{n-2,s-1} + A_{n-2,s})$$

and therefore if

$$B_{n,s}(x-n) - B_{n-2,s-1}N$$

that is, if

$$(x-n) \left[B_{n,s} - B_{n-2,s} - (2n-2)(n-1) \right]$$

$$+ \left[B_{n,s} - (2s+1)B_{n-2,s} \right]$$

But this is the case
and the other similar

The verification will

PAINVIN (1858, February).

[Sur un certain système d'équations linéaires. *Journ. de Liouville* (2), iii. pp. 41-46.]

The system of equations referred to in the title of Painvin's paper had presented themselves to Liouville in the course of the research which led to his "Mémoire sur les transcendantes elliptiques . . ." (*Journ. de Liouville* (1), v. pp. 441-464). Painvin's reason for taking up the subject was his belief that one of Liouville's results could be more simply arrived at by the use of determinants; and in a few lines of introduction he succeeds in showing that the result in question can be viewed as merely the resolution of the determinant

$$\begin{vmatrix} r & a & . & . & \dots & . & . \\ n(a-1) & r-1 & 2a & . & \dots & . & . \\ . & (n-1)(a-1) & r-2 & 3a & \dots & . & . \\ . & . & (n-2)(a-1) & r-3 & \dots & . & . \\ \dots & \dots & \dots & \dots & r-n+1 & na & \\ . & . & . & . & . & a-1 & r-n \end{vmatrix}$$

into factors.

In explanation of the process followed the case of the fourth order

$$\begin{vmatrix} r & a & . & . \\ 3(a-1) & r-1 & 2a & . \\ . & 2(a-1) & r-2 & 3a \\ . & . & a-1 & r-3 \end{vmatrix}$$

will suffice. Increasing each element of the first row by the corresponding elements of the other rows,—an operation which we may for the nonce symbolise by

$$\text{row}_1 + \text{row}_2 + \text{row}_3 + \dots,$$

—he removes the factor $r + 3a - 3$ and finds left the cofactor

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 3(a-1) & r-1 & 2a & . \\ . & 2(a-1) & r-2 & 3a \\ . & . & a-1 & r-3 \end{vmatrix}.$$

On this are performed the operations
 $col_1 - col_2, col_2 - col_1$
 the result being a determinant

$$- \left| \begin{array}{c} 3\alpha - r - 2 \\ 2 - 2\alpha \end{array} \right|$$

Finally, after changing the operations

$row_1 + row_2 + row_3 + \dots$,
 are performed, the result

$$\left| \begin{array}{c} r - \alpha \\ 2(\alpha - 1) \end{array} \right|$$

being a determinant exactly with $r - \alpha$ instead of r . This is formed into

$$(r + \alpha - 2) \left| \begin{array}{c} \end{array} \right|$$

and so on.

The value thus obtained for the $(n+1)^{th}$ order is

$(r + na - n)(r + na - n - 2\alpha + 1)(r + na - n - 2\alpha + 2) \dots$
 each factor being less than the previous one, and the result a function of $\alpha(\alpha - 1)$.

The special case is noted where the $n+1$ resulting factors are alike.

$$\left| \begin{array}{ccccc} r & \frac{1}{2} & 2 & & \\ \frac{n}{2} & r - 1 & \frac{1}{2} & & \\ -\frac{n-1}{2} & r - 2 & \frac{3}{2} & & \\ -\frac{n-2}{2} & r - 3 & & & \\ \dots & \dots & & & \end{array} \right|$$

but a preferable is, evidently,

$$\left| \begin{array}{cccccc} p & 1 & . & . & . & . & . \\ -n & p-2 & 2 & . & . & . & . \\ -n+1 & p-4 & 3 & . & . & . & . \\ -n+2 & p-6 & . & . & . & . & . \\ & & p-2n+2 & n & & & \\ & & . & . & . & . & \\ & & -1 & p-2n & & & \end{array} \right| = (p-n)^{n+1}.$$

HEINE, E. (1858, Sept.).

[Auszug eines Schreibens über die Laméschen Functionen an den Herausgeber. Einige Eigenschaften der Laméschen Functionen. *Crell's Journ.*, lvi. pp. 79-86, 87-99.]

In the case of Heine the functions afterwards known as "continuants" made their appearance under totally different circumstances, viz., while he was engaged in transforming a special homogeneous function of the second degree by means of an orthogonal transformation. It will be remembered that if the quadric

$$\begin{aligned} a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \dots \\ + a_{22}x_2^2 + 2a_{23}x_2x_3 + \dots \\ + a_{33}x_3^2 + \dots \end{aligned}$$

be transformed by an orthogonal transformation into

$$A_{11}\xi_1^2 + A_{22}\xi_2^2 + A_{33}\xi_3^2 + \dots$$

the coefficients of the latter expression are the roots of the equation

$$\left| \begin{array}{cccccc} a_{11}-A & a_{12} & a_{13} & . & . & . \\ a_{12} & a_{22}-A & a_{23} & . & . & . \\ a_{13} & a_{23} & a_{33}-A & . & . & . \\ & & & . & . & . \\ & & & . & . & . \end{array} \right| = 0.$$

Now Heine's peculiar quadric was

$$\begin{aligned} c_0^2x_0^2 - 2\kappa c_0c_1x_0x_1 \\ + (c_1^2 + c_2^2)x_1^2 - 2\kappa c_2c_3x_1x_2 \\ + (c_3^2 + c_4^2)x_2^2 - \\ + (c_{2\sigma-1}^2 + c_{2\sigma}^2)x_{\sigma}^2 \end{aligned}$$

where in every case the coefficient vanishes if their suffixes differ by

$$c_0^2 = \frac{1}{2}(n)(n+1),$$

$$c_1^2 = \frac{1}{2}(n-1)(n+1)$$

THE BOSTONIAN

$$c_r^2 = \frac{1}{4}(n-r)(n-r-1)$$

• • • • •

$$c^2_{n-1} = \frac{1}{2}n,$$

$$\text{and } \kappa = \frac{c^2 - b^2}{c^2 + b^2}.$$

s naturally led to the

He was thus naturally led to the

where either $c_{2\sigma}^2$ is c_{n-1}^2 , or $c_{2\sigma}^2$ is c_n^2 .
 From a knowledge of Painvin's
 side of the equation as being
 fraction

$$\text{fraction} \quad z - c_0^2 - \frac{\kappa c_0^2}{z - c_1^2}$$

but he ventured nothing in
 case where $b=0$ and where
 at the time too troublesome
 this case the continued frac

$$z(z - 2^2)(z - 4^2)$$

$$= \frac{z(z - 2^2)(z - 4^2)}{(z - 1^2)(z - 3^2)}$$

$$\text{and } \frac{(z-1^2)(z-3^2)/z-}{(z-2^2)(z-4^2)}$$

for his words are—"Einen
des Kettenbruchs habe ich zu

SCHLÄFLI, L. (1858).

[On the multiple integral $\int dx dy \dots dz$ whose limits are $p_1 = a_1 x + b_1 y + \dots + h_1 z > 0, p_2 > 0, \dots, p_n > 0$, and $x^2 + y^2 + \dots + z^2 > 1$. *Quart. Journ. of Math.*, ii. pp. 269-301, iii. pp. 54-68, 97-108.]

The determinant which makes its appearance in the course of Schläfli's research is

$$\begin{vmatrix} 1 & -\cos \alpha & & & & & \\ -\cos \alpha & 1 & -\cos \beta & & & & \\ & -\cos \beta & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & -\cos \eta & \\ & & & & & -\cos \eta & 1 & -\cos \theta \\ & & & & & & -\cos \theta & 1 \end{vmatrix}$$

which for shortness' sake he denotes by

$$\Delta(\alpha, \beta, \gamma, \dots, \eta, \theta)$$

and whose connection with continued fractions he therefore specifies by the equation

$$\frac{\Delta(\alpha, \beta, \gamma, \dots, \eta, \theta)}{\Delta(\beta, \gamma, \dots, \eta, \theta)} = 1 - \frac{\cos^2 \alpha}{1} - \frac{\cos^2 \beta}{1} - \dots - \frac{\cos^2 \eta}{1 - \cos^2 \theta}.$$

The first property noticed is, naturally,

$$\Delta(\alpha, \beta, \gamma, \dots, \theta) = \Delta(\beta, \gamma, \dots, \theta) - \cos^2 \alpha \cdot \Delta(\gamma, \dots, \theta).$$

Later there is given what may be viewed as an extension of this, viz.,

$$\begin{aligned} \Delta(\alpha, \dots, \delta, \epsilon, \zeta, \eta, \theta, \dots, \lambda) &= \Delta(\alpha, \dots, \delta, \epsilon) \cdot \Delta(\eta, \theta, \dots, \lambda) \\ &\quad - \cos^2 \zeta \cdot \Delta(\alpha, \dots, \delta) \cdot \Delta(\theta, \dots, \lambda), \end{aligned}$$

the proof being said to present no difficulty. The third is a little more complicated, and is logically led up to by taking four instances of the first property, viz.,

$$\begin{aligned} \Delta(\alpha, \beta, \gamma, \dots, \zeta) &= \Delta(\beta, \gamma, \dots, \zeta) - \cos^2 \alpha \cdot \Delta(\gamma, \delta, \dots, \zeta), \\ \Delta(\beta, \gamma, \delta, \dots, \zeta, \eta) &= \Delta(\gamma, \delta, \dots, \eta) - \cos^2 \beta \cdot \Delta(\delta, \dots, \zeta, \eta), \\ \Delta(\gamma, \delta, \dots, \zeta, \eta, \theta) &= \Delta(\gamma, \delta, \dots, \eta) - \cos^2 \theta \cdot \Delta(\gamma, \delta, \dots, \zeta), \\ \Delta(\delta, \dots, \zeta, \eta, \theta, \alpha) &= \Delta(\delta, \dots, \zeta, \eta, \theta) - \cos^2 \alpha \cdot \Delta(\delta, \dots, \zeta, \eta), \end{aligned}$$

using in connection with these the

$$\Delta(\delta, \dots, \zeta, \eta), - \Delta(\delta, \dots, \zeta)$$

respectively, performing addition hand sum vanishes, the result

$$\Delta(\alpha, \beta, \gamma, \delta, \dots, \zeta) \cdot \Delta(\delta, \dots, \zeta, \eta) \\ = \{ \Delta(\beta, \gamma, \delta, \dots, \zeta, \eta) -$$

The fourth property concerns

$$\left| \begin{array}{l} \Delta(\beta, \gamma, \dots, \eta, \theta) \\ \Delta(\beta, \gamma, \dots, \eta) \end{array} \right.$$

which by reason of the first is

$$\left| \begin{array}{l} \Delta(\beta, \gamma, \dots, \eta, \theta) \\ \Delta(\beta, \gamma, \dots, \eta) \end{array} \right.$$

or

$$\left| \begin{array}{l} \Delta(\gamma, \dots, \eta, \theta) \\ \Delta(\gamma, \dots, \eta) \end{array} \right.$$

and ultimately, "by repeating to

$$\cos^2 \alpha \cos$$

If we use for a moment
viz., where

$$a_1 + \frac{b_1}{a_2} + \frac{b_2}{a_3}$$

Schlafli's results are seen

$$K(\beta_1, \beta_2, \beta_3, \dots) =$$

$$K(\beta_1, \beta_2, \dots, \beta_k, \beta_{k+1}, \dots)$$

$$- \beta_k K($$

$$\left. \begin{aligned} & K\left(\begin{smallmatrix} \beta_1 & \beta_{n-2} \\ 1 & 1 \dots 1 \end{smallmatrix}\right) \cdot K\left(\begin{smallmatrix} \beta_4 & \beta_{n+1} \\ 1 & 1 \dots 1 \end{smallmatrix}\right) \\ & K\left(\begin{smallmatrix} \beta_1 & \beta_n & \beta_1 \\ 1 & 1 \dots 1 & 1 \end{smallmatrix}\right) \cdot K\left(\begin{smallmatrix} \beta_2 & \beta_{n-1} \\ 1 & 1 \dots 1 \end{smallmatrix}\right) \end{aligned} \right\} = \left. \begin{aligned} & K\left(\begin{smallmatrix} \beta_2 & \beta_{n-1} \\ 1 & 1 \dots 1 \end{smallmatrix}\right) \\ & - K\left(\begin{smallmatrix} \beta_3 & \beta_n \\ 1 & 1 \dots 1 \end{smallmatrix}\right) \end{aligned} \right\} K\left(\begin{smallmatrix} \beta_4 & \beta_{n-2} \\ 1 & 1 \dots 1 \end{smallmatrix}\right), \\ K\left(\begin{smallmatrix} \beta_2 & \beta_n \\ 1 & 1 \dots 1 \end{smallmatrix}\right) \cdot K\left(\begin{smallmatrix} \beta_1 & \beta_n \\ 1 & 1 \dots 1 \end{smallmatrix}\right) \\ K\left(\begin{smallmatrix} \beta_1 & \beta_{n-1} \\ 1 & 1 \dots 1 \end{smallmatrix}\right) \cdot K\left(\begin{smallmatrix} \beta_1 & \beta_{n-1} \\ 1 & 1 \dots 1 \end{smallmatrix}\right) \end{aligned} \right\} = (-1)^n \beta_1 \beta_2 \beta_3 \dots \beta_n, \end{math>$$

the only change being the writing of β_1, β_2, \dots for $-\cos^2\alpha, -\cos^2\beta, \dots$

WORPITZKY (1865, April).

[Untersuchungen über die Entwicklung der monodromen und monogenen Functionen durch Kettenbrüche. (Sch. Progr.) 39 pp., Berlin.]

Of the six sections into which the paper giving the results of Worpitzky's painstaking investigation is divided it is only the first headed "Fundamentalrelationen" which concerns us, these relations being nothing else than what we should now call "properties of continuants."

He takes his continued fraction in the same form as Schlafli, viz.,

$$1 + \frac{a_1}{1} + \frac{a_2}{1} + \dots + \frac{a_n}{1},$$

showing of course that it equals

$$\frac{N_{1,n}}{N_{2,n}},$$

where

$$N_{s,n} = \begin{vmatrix} 1 & 1 & . & . & . & . & . \\ -a_n & 1 & 1 & . & . & . & . \\ . & -a_{n-1} & 1 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & -a_{n-1} & 1 \\ . & . & . & . & . & . & -a_n \end{vmatrix}.$$

The first matter of interest is the ~~ex~~ products of a_k, a_{k-1}, \dots, a_n , e.g.

$$N_{1,8} = 1 + (\alpha_1 + \alpha_2 +$$

This is written in the form

$$1 + a_{n,n}^{-1} + a_{n,n}^{-2} =$$

where, he says, " $a_{\kappa,n}$ die Summe alle aufgefassten) Combinationscomplexionen deutet, welche sich aus a_{κ} , $a_{\kappa+1}$, . . . , bilden lassen, dass nicht zwei neben ein a_{κ} , $a_{\kappa+1}$ dieser Reihe in den einzelnen kommen." By way of proof it is pointed out that $a_{\kappa,n}$ is independent of all the a 's is

$$\left| \begin{array}{cccccc} 1 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & 0 & 1 & 1 & \\ & & & & 0 & 1 & \end{array} \right\} i.e.$$

(2) that the cofactor* of $(-\alpha_r)(-\alpha_s)(-\alpha_t)$
the α 's are consecutive is

$$\begin{array}{ccccccccc}
 1 & 1 & & & & & & & \\
 0 & 1 & 1 & & & & & & \\
 \cdot & \cdot & \cdot & \cdot & & & & & \\
 0 & 1 & 1 & 0 & & & & & \\
 1 & 0 & 0 & 1 & 0 & & & & \\
 1 & 0 & 0 & 1 & 0 & 0 & & & \\
 0 & 1 & 1 & 0 & 1 & 0 & 1 & & \\
 \cdot & & \\
 0 & 1 & & & & & & & \\
 0 & & & & & & & & \\
 \end{array}$$

* To obtain the cofactor of the product of a number a determinant in the said product, 0's in all the other places occurring in the determinant Worpitzky puts a 1 in the determinant in these elements belong, and 0's for all the other elements.

and (3) that the cofactor of $(-a_r)(-a_s)(-a_t) \dots$ when no two of the a 's are consecutive and their number is p , is

$$\begin{array}{cccccc} 1 & 1 & & & & \\ 0 & 1 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & 0 & 1 & 1 & & \\ & 1 & 0 & 0 & & \\ & 0 & 1 & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & 0 & 1 & 1 & \\ & & 1 & 0 & 0 & \\ & & 0 & 1 & 1 & \\ & & & \cdot & \cdot & \cdot \\ & & & 0 & 1 & 1 & \\ & & & & 0 & 1 & \end{array}$$

$$\text{i.e. } \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^p \text{ i.e. } (-1)^p,$$

and that, therefore, the cofactor of $a_r a_s a_t \dots$ in this case is + 1. In exactly similar fashion by partitioning $N_{k,n}$ into terms which contain $-a_s$ and terms which do not, he finds

$$\text{where } N_{k,n} = D_0 - a_s D_s,$$

$$D_0 = \begin{vmatrix} 1 & 1 & & & & \\ -a_k & 1 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & -a_{s-1} & 1 & 1 & \\ & & 0 & 1 & 1 & \\ & & -a_{s+1} & 1 & 1 & \\ & & & \cdot & \cdot & \\ & & & & -a_{n-1} & 1 & 1 \\ & & & & & -a_n & 1 & \end{vmatrix},$$

$$= \begin{vmatrix} 1 & 1 & & & & \\ -a_k & 1 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & -a_{s-2} & 1 & 1 & \\ & & -a_{s-1} & 1 & 1 & \end{vmatrix} \begin{vmatrix} 1 & 1 & & & & \\ -a_{s+1} & 1 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & -a_{n-2} & 1 & 1 & \\ & & -a_{n-1} & 1 & 1 & \\ & & & & -a_n & 1 & \end{vmatrix} = N_{k,s-1} \cdot N_{s+1,n+1}$$

$$D_s = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 0 & 0 & \dots \\ -a_k & 1 & 1 & \dots & 1 & 0 & 0 & \dots \\ \dots & \dots \\ -a_{s-2} & 1 & 1 & \dots & 1 & 0 & 0 & \dots \\ -a_{s-1} & 0 & 0 & \dots & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots & -a_{s+1} & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots & -a_{s+2} & 0 & 0 & \dots \end{vmatrix}$$

$$N_{k,n} = N_{k,n-1}N_{s+1,n} + a_s N_{k,s-2}N_{s+2,n}$$

already obtained in a different way by Schläfli.
Lastly, taking a determinant of the same form a
having

$$-a_s, -a_{s-1}, \dots, -a_{s+1}, -a_s, -a_k, -a_{k+1}, \dots, -a_{n-1}$$

for its minor diagonal of a 's, he obtains for it by isolating a_k

$$N_{s,k+1}N_{k,n} + a_k N_{s,k+2}N_{s+1,n},$$

and by isolating the second a_k

$$N_{s,k}N_{k+1,n} + a_k N_{s,k+1}N_{s+2,n};$$

and thus deduces

$$N_{k,n}N_{k+1,n} - N_{k,s}N_{s+1,n} = -a_k(N_{s+1,n}N_{s+2,n} - N_{s+1,n}N_{s+2,n}).$$

It is then noted that the bracketed expression on the right diff...

from the expression on the left merely in having $k+1$ in place of k ; so that there results

$$\begin{aligned} N_{k,n}N_{k+1,n} - N_{k,n}N_{k+1,n} &= (-1)^2 a_k a_{k+1} (N_{k+2,n}N_{k+3,n} - N_{k+2,n}N_{k+3,n}) \\ &= \dots \dots \dots \\ &= (-1)^{s-k+1} a_k a_{k+1} \dots a_{s+1} N_{s+3,n}. \end{aligned}$$

This also, it will be seen, is connected with a result of Schläfli's; for putting $s=n-1$ we have*

$$\begin{vmatrix} N_{k+1,n-1} & N_{k,n-1} \\ N_{k+1,n} & N_{k,n} \end{vmatrix} = (-1)^{n-k} a_k a_{k+1} \dots a_n,$$

which becomes identical with Schläfli's last proposition on transposing the two rows of the determinant and (what is equally immaterial) putting $k=1$.

THIELE, T. N. (1869, 1870).

[Bemærkninger om Kjædebrøker. *Tidsskrift for Math.* (2), v. pp. 144-146.

Den endelige Kjædebrøksfunktions Theori. *Tidsskrift for Math.* (2), vi. pp. 145-170.]

The first of the two notes comprising Thiele's first paper contains only one result, viz.,

$$a_1 + \frac{b_1}{a_2} + \frac{b_2}{a_3} + \dots + \frac{b_n}{a_n} = \frac{(a_1, a_2, \dots, a_n)}{(a_2, \dots, a_n)},$$

where (a_1, a_2, \dots, a_n) is used to stand for

$$\begin{vmatrix} a_1 & b_1 & & \dots & & & \\ -1 & a_2 & b_2 & \dots & & & \\ & \dots & \dots & \dots & \dots & & \\ & & & a_{n-1} & b_{n-1} & & \\ & & & \dots & & & \\ & & & -1 & a_n & & \end{vmatrix}$$

* In giving to $N_{s+1,n}$, $N_{s+2,n}$, $N_{s+3,n}$ the values 1, 1, 0 which are necessitated by assuming the generality of the recursion-formula

$$N_{k,n} = N_{k+1,n} + a_k N_{k+2,n}$$

Worpitzky forgets to note that in these cases the proposition $N_{k,n} = N_{n,k}$, used by him in the demonstration, does not hold.

There is nothing to indicate the discovery, notwithstanding the fact taining virtually the same identity

The other paper may be described continued fractions with the help of b_1, b_2, \dots are used a_{12}, a_{23}, \dots

$$\begin{array}{cccccc} a_p & & a_{p,p+1} & & \cdot & \\ 1 & & a_{p+1} & & a_{p+2} & \\ | & & | & & | & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ | & & | & & | & \\ \cdot & & \cdot & & \cdot & \end{array}$$

is denoted by

K

Further, this determinant is sometimes called a "K-Determinant," or, shortly, a "K-Det." (See the paper in the Proceedings of the Royal Society, Vol. 149-152) is devoted to a stater

There is no need to rehearse the section being alone that when we have to do with the double use of a previous

$$\begin{aligned} K(h, m) &= K(h, k-1) \cdot K(k, m) \\ K(h, n) &= K(h, k-1) \cdot K(k, n) \end{aligned}$$

where h, k, m, n are in ascending order, and we obtain $K(h, k-1)$ and obtain $K(k, m)$ and $K(k, n)$.

$$\begin{vmatrix} K(h, m) & K(k, m) \\ K(h, n) & K(k, n) \end{vmatrix} = a_{k-1, k} \cdot K$$

Then by taking the particular place of h and $k+1$ in place

$$\begin{vmatrix} K(k, m) & K(k+1, m) \\ K(k, n) & K(k+1, n) \end{vmatrix}$$

which when applied to one of the

$$\begin{vmatrix} K(k, m) & K(k+1, m) \\ K(k, n) & K(k+1, n) \end{vmatrix} = a_{k, k+1}$$

$$\begin{vmatrix} K(k, m) & K(k+1, m) \\ K(k, n) & K(k+1, n) \end{vmatrix} = a_{k, k+1}$$

and finally

$$= a_{k,k+1} a_{k+1,k+2} \dots a_{m,m+1} \cdot \begin{vmatrix} K(m+1,m) & K(m+2,m) \\ K(m+1,n) & K(m+2,n) \end{vmatrix},$$

$$= a_{k,k+1} a_{k+1,k+2} \dots a_{m,m+1} \cdot K(m+2,n). \quad (\beta)$$

Further, by using this to make a substitution in the previous result
(a) there is obtained

$$\begin{vmatrix} K(h,m) & K(k,m) \\ K(h,n) & K(k,n) \end{vmatrix} = a_{k-1,k} a_{k,k+1} \dots a_{m,m+1} \cdot K(h,k-2) K(m+2,n), \quad (\gamma)$$

which on putting $k = h + 1$ and $m = n - 1$ becomes

$$\begin{vmatrix} K(h,n-1) & K(h+1,n-1) \\ K(h,n) & K(h+1,n) \end{vmatrix} = a_{h,h+1} a_{h+1,h+2} \dots a_{n-1,n},$$

—a result which may be compared with one of Schläfli's and Worpitzky's, but which is more general in that the main diagonal of each "K-Determinant" does not consist of units.

LIST OF AUTHORS

whose writings are herein dealt with.

	PAGE		PAGE
1853. SYLVESTER	130	1857. CAYLEY. . . .	144
1853. SPOTTISWOODE	134	1858. PAINVIN	147
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(Issued separately, February 26, 1904.)

On the Origin of the Epiphysial Structure in the Chick.
(Edin.), M.R.C.S. (Eng.), C^a
Anatomy, United College, U^m
municated by Dr W. G. A.

(MS. received January 4,

CONT

- (1) RESULTS OF THE PRESENT RESEARCH
- (2) COMPARISON OF RESULTS
- (3) SUMMARY AND CONCLUSIONS
- (4) LITERATURE
- (5) EXPLANATION OF ILLUSTRATIONS

(1) RESULTS OF THE

A number and 60th hours of incubation difficult in every instance to nature of the epiphysis, still condition was distinctly marked demonstrating in all cases the condition will be explained later.

Fig. 1 is drawn from a chick-embryo at the 50th hour of incubation, and represents a transverse section of the thalamencephalon in the pineal region. The larger of the two evaginations lies distinctly to the left of the mesial plane (which is represented by the dotted line in the figure), while on the right side a much

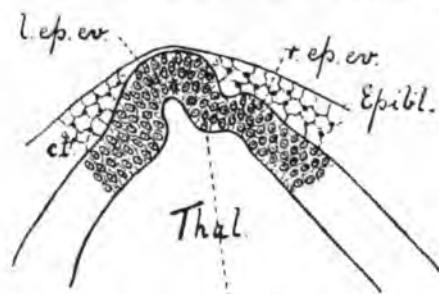


FIG. 1.

smaller evagination exists. The latter was found to be evident in the whole series of sections of the pineal region in this embryo, but it was in every instance much smaller than the left evagination.

Fig. 2 is from a chick-embryo at the 60th hour of incubation,

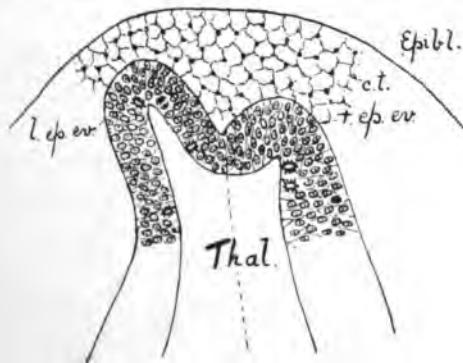


FIG. 2.

and represents a transverse section of the roof of the thalamencephalon in the pineal region as in the previous instance. The resemblance between this fig. and the fig. No. 5 which illustrates Dendy's paper (11) is most striking, as will be at once recognised on comparing the two. Fig. 2 shows with marked clearness the simul-

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taneous presence of both outgrowths. Here, again, two; but the right outgrowth in the previous instance (to afford distinct proof of arises in the form of two of this early stage could have given ample demonstration epiphysial outgrowths. I outgrowths the left was b

A study of the later stages of the chick shows that the brief—the right and left are found to form the single outgrowth. This takes place at the 60th hour of incubation, observed between the 50th and 60th hours. It has a very transient existence, which probably explains why the condition is not often recognised. But it should be noted that it is a right or smaller evagination, which is evanescent, so that it was quite altogether (more especially instead of a series), or to the cerebral wall due to faulty fixation. All research were incubated under the same conditions of those anomalies which occurred when the incubator was avoided. A Biles' fluid, which is an excellent fixative, was used, and all risks of shrinkage were thus avoided.

As has been already stated, the epiphysis ceases to exist after a certain period; but one cannot draw any definite conclusion regarding the duration of the condition. The recognised fact that chick-embryos of growth. In some cases, a condition was observed previously to the appearance of the

while in the other cases this condition was distinctly evident after the 60th hour of incubation.

Fig. 3 is, like the others, a transverse section of the thalamencephalon in the pineal region, and is from a chick-embryo at the end of the 3rd day of incubation. This figure represents what might be termed the unpaired condition of the epiphysis. On close examination, however, the presence of two small angular recesses within the evagination will be noted, and it may be suggested that these are probably lingering evidences of the previously existing bilateral outgrowths—the process of coalescence having apparently just taken place.

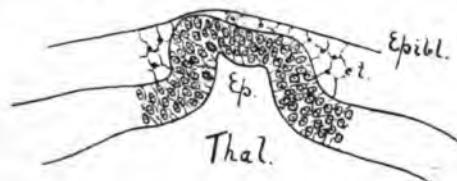


FIG. 3.

It therefore appears that what in its earlier stages of development used to be looked upon as a mesially placed epiphysial evagination is really situated to the left of the mesial plane, while a more feebly formed evagination exists on the right side. This bilateral condition is, however, very transitory, and soon gives rise to the unpaired condition of the epiphysis by a coalescence of the primary elements.

(2) COMPARISON OF RESULTS.

The results of this research are of interest in so far as they support the observations previously made by the author in the Amphibia (8 and 9). They also agree in the main with the results obtained by various observers in reference to other classes of the Vertebrata. In Amphibia the author has described the presence in the early stages of right and left recesses from the roof of the thalamencephalon, of which the left is the better developed of the two; and has shown that these very soon coalesce to form a single epiphysial structure. It will be at once observed that these conclusions are corroborated in the case of the chick.

It is also interesting to compare the results of the present research with those of Dendy (11) on *Hatteria*. This observer has

demonstrated in embryos of left epiphysial outgrowths, each other. Of these, the rise to the pineal eye, which into anything resembling to the roof of the thalamus stalk. So also in the chick of the two. It is, however, right evagination, and the

Hill (17) has described Teleosteans and in *Amia* the right outgrowth was while they showed no te

Locy (19) has been a He describes the epiphysial pair of united accessory therefore, the paired eyes in the case of the chick

This research was conducted at the United College, University of Cambridge, my appointment both as Fellow of St Andrew's College and best thanks to Professor Locy, which were afforded me the opportunity to intend to study the embryology of the Mammalia in order to compare the lateral condition of the Vertebrates.

(2)

(1) The epiphysial outgrowth in the form of right and left epiphyses, the left is the larger.

* My attention was first drawn to the study of *Varis* by Dr. J. C. Gray, who like other structures of the head, coalesce. This is described on page 458 of the above work, the worker-bee having the mesial plate.

(2) The right primary evagination blends with the left at an early stage of development to form a unified structure.

(3) These observations correspond for the most part with those already made by the author in the case of the Amphibia. They also agree in many ways with those of Béraneck, Dendy, Gaskell, Hill and Lacy in other classes of the Vertebrata. As a result of this, it is evident that in the four lower Vertebrate classes the epiphysis cerebri arises as a bilateral, and not as a mesial structure.

(4) It is probable that the ancestors of Vertebrates possessed a pair of parietal eyes, and not a single unpaired structure.

(4) LITERATURE.

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(5) EXPLANATION OF FIGURES.

[The figures were drawn with the aid of Zeiss's camera lucida apparatus. Zeiss's objective A and ocular No. 3 were employed.]

ct., subcutaneous connective tissue; *ep.*, epiphysis; *epib.*, epiblast; *l. ep. ev.*, left epiphysial evagination; *r. ep. ev.*, right epiphysial evagination; *thal.*, cavity of thalamencephalon.

Fig. 1. Transverse section of the roof of the thalamencephalon in the pineal region. Embryo-chick at the 50th hour of incubation. The right and left primary epiphysial evaginations are seen. Two germinal nuclei in a condition of karyokinesis are observable. The dotted line represents the mesial plane.

Fig. 2. Transverse section of the roof of the thalamencephalon in the pineal region. Embryo-chick at the 60th hour of incubation. The right and left primary epiphysial evaginations are especially well marked. Several germinal nuclei are seen. The mesial plane is represented by the dotted line.

Fig. 3. Transverse section of the roof of the thalamencephalon in the pineal region. Embryo-chick at the end of the 3rd day of incubation. The unpaired condition of the epiphysis is shown. The presence of two small angular recesses, however, within the epiphysial evagination may denote traces of the previously existing bilateral condition.

Issued separately March 17, 1904.]

**Theorem regarding the
of a Quadric. B**

(MS. received July 27, 1

(1) The theorem in quest
several passages in Jacobi's
orthogonal transformation.

which simultaneously change

$$x_1^2 + x_2^2 + \dots + x_n^2$$

and

$$\sum_{\kappa\lambda} a_{\kappa\lambda} x_{\kappa} x_{\lambda} \text{ into}$$

Jacobi proceeds to show (p.

$$G_1^p y_1^2 + G_2^p$$

where p is any positive i
 x_1, x_2, \dots, x_n (" expressi
licet"). The actual result
(p. 14) he reaches a theore
the restriction on p so as to
but the opportunity is n
neglect probably is that 1
subject when prepared to
this may be, certain it is
hypothetical form of the
words (p. 20) are :—

His words $G_1^p y_1^2 + G_2^p$

$$G_1^p a_{1\kappa} a_{1\kappa} + G_2^p a_{2\kappa}$$

and where, we may add, th
and where, we may add, th
tion. Regarding the vali

* JACOBI, C. G. J., De binis
ordinis per substitutiones lin
Crelle's Journ., xii. pp. 1-69. (

aduced to show that whether p be a positive or negative integer the coefficient of $x_k x_\lambda$ is a rational function of the coefficients of the original quadric.

With this general statement of the case before us, let us take up the individual results in order, and see what is obtainable therefrom in the light of later work.

(2) The primary result is the transformation implied in the equation

$$\sum_{\kappa\lambda} a_{\kappa\lambda} x_\kappa x_\lambda = G_1 y_1^2 + G_2 y_2^2 + \dots + G_n y_n^2.$$

This, for our purpose, it is essential to write in a form which brings into evidence the matrix M of the discriminant of the quadric, viz., in the form

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ a_{11} & a_{12} & a_{13} & x_1 \\ a_{21} & a_{22} & a_{23} & x_2 \\ a_{31} & a_{32} & a_{33} & x_3 \end{array} = G_1 y_1^2 + G_2 y_2^2 + G_3 y_3^2,$$

where, merely for shortness' sake, only three variables are taken. Now, as Jacobi himself showed, any equation which holds between the x 's and y 's will still hold if we put

$$\left(\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right) (x_1, x_2, x_3) \text{ for } x_1, x_2, x_3$$

and

$$G_1 y_1, G_2 y_2, G_3 y_3 \text{ for } y_1, y_2, y_3$$

This substitution, however, in the bipartite function on the left results simply in the matrix of the discriminant being twice multiplied by itself,* so that we have

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ M^3 & & & x_1 \\ & & & x_2 \\ & & & x_3 \end{array} = G_1^3 y_1^2 + G_2^3 y_2^2 + G_3^3 y_3^2.$$

* *Trans. R. S. Edinb.*, xxxii, p. 480.

It is to be noted that the use of polarisation for measuring the light intensities is not prohibited of this appliance, even when the light is polarised passed through the latter. No change of polarisation of polarisation can be caused by the entrance the glass, for that only takes place normally to the If the two beams are plane polarised vertically before entrance, with a view to the adjustment intensities later on by means of a Nicol prism, the plane of polarisation in each case is either in a plane of incidence on the silver surfaces no change can occur. On the other hand, if the two beams respective planes of polarisation inclined to the vertical, these beams, because they are each to parallel silver surfaces, will emerge plane polarised the plane of polarisation of each has been rotated. Hence in both cases the analysing Nicol can be used for the purpose of measuring the light intensity, will have been permanently displaced through a

A suggested application of the juxtapositor device be readily understood by anyone conversant with Laurent's "half-shade" polarimeter. In this parallel beams of light polarised in planes another are passed through a substance whose desired to measure, and are then analysed by a prism. By properly adjusting the position of the half-circles of light seen in the eyepiece of the instrument, the two beams, can be made equally bright. In this way a much more accurate setting of the instrument obtained than when, as in the ordinary case, light is employed and the Nicol is set to zero. The accuracy of the measurement in Laurent's instrument turns on the degree of precision with which it is determined when the two halves of the circle are equally bright. Now, these two halves are a dark line, and according to, as explained in

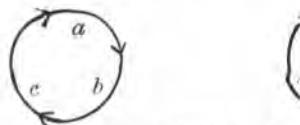
The Three-line Determination By Thomas H. Huxley

(Second copy of MS)

y of MS. A.

(1) If the array in question has

its score of three-line determinants may be viewed as consisting of two sets, each of the first set containing at least $|a_1 b_2 c_3|$, and each of the second set from $|f_1 g_2 h_3|$. Further, either set consisting of one unique member and members each, the members of a sub-set another by performing the cyclical substitution



In this way a convenient notation for the will be found to be

$$\left\{ \begin{array}{l} a_1b_2c_3 \\ a_1b_2g_8 \\ a_1b_2g_3 \\ a_1b_2g_3 \end{array} \right\} \equiv \left\{ \begin{array}{l} a_1b_2c_3 \\ b_1c_2g_8 \\ b_1c_2g_3 \\ b_1c_2f_3 \end{array} \right\} \equiv \left\{ \begin{array}{l} c_1a_2b_2 \\ c_1a_2g_8 \\ c_1a_2g_3 \\ c_1a_2f_3 \end{array} \right\} \equiv \left\{ \begin{array}{l} 1 \\ 4 \\ 7 \\ 7 \end{array} \right\}$$

$$\left| \begin{array}{c} f_1g_2h_3 \\ \alpha_1f_2g_3 \\ \alpha_2f_3g_1 \\ \alpha_3f_1g_2 \end{array} \right|, \quad \left| \begin{array}{c} f_1g_2h_3 \\ \alpha_1f_2g_3 \\ \alpha_2f_3g_1 \\ \alpha_3f_1g_2 \end{array} \right|, \quad \left| \begin{array}{c} b_1f_2g_3 \\ b_2f_3g_1 \\ b_3f_1g_2 \\ b_4f_4g_1 \end{array} \right|, \quad \left| \begin{array}{c} b_1f_2g_3 \\ b_2f_3g_1 \\ b_3f_1g_2 \\ b_4f_4g_1 \end{array} \right| \right\} = \left\{ \begin{array}{c} 1', 2' \\ 4', 5' \\ 7', 8' \end{array} \right\}$$

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* The original MS. was ~~destroyed~~ lost in transit through the post.—[3]
20th March 1904, but was ~~destroyed~~ lost in transit through the post.—[3]

That this must in all cases be true, that the interchanges, which are the products of the first triad, are

$(f)_a$.

$(f)_b$.

$(f)_c$.

and that these when taken in combination with the interchanges of the second triad, which need to be performed on 00

second triad.

(4) The ten groups of such sets of interchanges are compactly exhibited as follows:—

00'

11'	44'	
88'	22'	5
66'	99'	32

11'

00'	- 44'	- 77'
- 66'	55'	22'
- 88'	33'	99'

22'

00'	- 55'	- 88'
- 44'	66'	33'
- 99'	11'	77'

44'

00'	- 22'	- 99'
- 11'	33'	55'
- 77'	66'	88'

55'

00'	- 33'	- 77'
- 22'	11'	66'
- 88'	44'	99'

77'

00'	- 11'	- 44'
- 33'	22'	88'
- 55'	99'	66'

88'

00'	- 22'	- 55'	
- 11'	33'	99'	
- 66'	77'	44'	

$$11' - 22' = 4$$

$$22' - 33' = 6$$

$$33' - 11' = 4$$

$$11' - 55' = 2$$

$$11' - 99' = 22$$

$$44' - 88' = 55$$

It will be seen that the second and the third from the second, and that the number of such triads is not so related: the cyclical substitutions of these would simply reproduce the

(6) It is interesting to note that two sets of three equivalents a fourth equivalent may be added. Thus for the seventh equivalent

$$\begin{vmatrix} |a_1b_2| & |a_2b_3| & | \\ |f_1g_2| & |f_2g_3| & | \\ |c_1h_2| & |c_2h_3| & |c \end{vmatrix}$$

for this can be shown equal to

$$\begin{vmatrix} |a_1b_2g_3| & |a_1b_2f_3| & | \\ |c_1h_2g_3| & |c_1h_2f_3| & | \end{vmatrix} \text{ i.e. } 3$$

and as the interchanges

$$\begin{pmatrix} a & b \\ f & g \end{pmatrix}, \begin{pmatrix} c & h \\ f & g \end{pmatrix}$$

alter only the sign of the three-line determinant, also be equal to

$$- \begin{vmatrix} |f_1g_2h_3| & |f_1g_2a_3| & | \\ |c_1h_2b_3| & |c_1h_2a_3| & | \end{vmatrix} \text{ i.e. } 3$$

and

$$- \begin{vmatrix} |a_1b_2h_3| & |a_1b_2f_3| & | \\ |f_1g_2h_3| & |f_1g_2f_3| & | \end{vmatrix} \text{ i.e. } -77$$

* The other similar interchanges $\begin{pmatrix} a & b \\ h & k \end{pmatrix}$ give nothing

be expressed in one and
other such products.

(9) We are thus pr
the two products who
this way as an equival
and apply our theore
repetition of the previo

$$|g_1b_2c_3| \cdot |c_1\alpha_2h_3| = \\ \text{i.e.} \qquad \qquad \qquad 23 = \\ \text{and}$$

$$|h_1b_2c_3| \cdot |c_1g_2\alpha_3| = \\ \text{i.e.} \qquad \qquad \qquad -59 =$$

It follows therefore th
each can only occur onc
the number of such is
arrangement of the thirty
those that form a triad,
that are complementary.

$$01' - 23 + \\ 02' - 31 + \\ 03' - 12 +$$

$$04' - 56 + \\ 05' - 64 + \\ 06' - 45 +$$

$$07' - 89 + \\ 08' - 97 + \\ 09' - 78 +$$

$$14' + 82' + \\ 25' + 93' + \\ 36' + 71' +$$

$$16' + 49' + 7 \\ 24' + 57' + 8 \\ 35' + 63' + 9$$

The Sum of the S
Determinant.

(MS. received July 25)

(1) The fundamental pro
signed primary minors of a

(A) An expression for the
minors of any determinant
next higher order whose first
minant for complementary
are units all positive or all n

(B) The sum of the signed i
possible as a determinant of
of the latter being the sum of i
the former, viz., the sum (r, s)

(C) If the elements of a de
quantity ω , the determinant
sum of its signed primary mi

(2) By the application of
are readily obtained—

The sum of the signed
| $a^0b^1c^2 \dots$ | is equal to 1

The sum of the signed P^r
order is equal to n times the
its variables.

Thus, the sum of the sig

$$= - \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & b & c \\ 1 & c & a & b \\ 1 & b & c & a \end{vmatrix} =$$

$$= - \begin{vmatrix} -3 & 1 & 1 & 1 \\ 1 & a & b & c \\ 1 & c & a & b \\ 1 & b & c & a \end{vmatrix} \div (a$$

$$= 3C(a, b, c) \div (a + b + c)$$

* Proceedings Roy. Soc.

zero elements of which are (obtain the equivalent expression $(7,7)\text{cof} + (6,7)(7,6)\text{cof} + (1,6)\text{cof}$)

i.e. $a_6 M \begin{pmatrix} b_1 & \dots & \\ a_1 & \dots & a_5 \\ c_1 & \dots & \end{pmatrix} -$

$$- c_5 \begin{vmatrix} 1 & a_1 & b_1 & \dots & \\ 1 & c_1 & a_2 & b_2 & \dots \\ 1 & \dots & c_2 & a_3 & b_3 \\ 1 & \dots & c_3 & a_4 & \dots \\ 1 & \dots & \dots & \dots & c_4 \end{vmatrix}$$

Of the two determinants h seen to be

$$= \begin{pmatrix} b_1 & \dots & \\ a_1 & \dots & a_4 \\ c_1 & \dots & \end{pmatrix} +$$

$$= \begin{pmatrix} b_1 & \dots & \\ a_1 & \dots & a_4 \\ c_1 & \dots & \end{pmatrix} - c_4 \begin{pmatrix} b_1 & \dots & \\ a_1 & \dots & a_3 \\ c_1 & \dots & \end{pmatrix}$$

and the second

$$= \begin{pmatrix} b_1 & \dots & \\ a_1 & \dots & a_4 \\ c_1 & \dots & \end{pmatrix} - b_4 \begin{pmatrix} b_1 & \dots & \\ a_1 & \dots & a_3 \\ c_1 & \dots & \end{pmatrix}$$

It thus follows that

$$M \begin{pmatrix} b_1 & \dots & \\ a_1 & \dots & a_6 \\ c_1 & \dots & \end{pmatrix} = \alpha_6 M \begin{pmatrix}$$

$$+ \begin{pmatrix}$$

(6) For the case of a
the b 's is 1 and each of
becomes

$$\begin{aligned}
 & (a_1 a_2 \dots a_n) \\
 & + (a_1 a_2 \dots a_{n-1}) \\
 & + (a_1 a_2 \dots a_{n-2}) \\
 & + (a_1 a_2 \dots a_{n-3}) \\
 & + (a_1 a_2 \dots a_{n-4}) \\
 & + \dots
 \end{aligned}$$

and therefore, like the
positive.

For example, the sum
continuant (a_1, a_2, a_3, a_4)

$$(a_1 a_2 a$$

$$\begin{aligned}
 & i.e. \\
 & a_1 a_2 a_3 + a_1 + a
 \end{aligned}$$

$$\begin{aligned}
 & i.e. \\
 & a_1 a_2 a_3 + a_1 a_2 a
 \end{aligned}$$

(7) If the expression
H's and their cofactors, i

$$\begin{aligned}
 & H_{n-1} + H_{n-2}(1, l) \\
 & + H_{n-3}(1, l) \\
 & + \dots
 \end{aligned}$$

which according to (VI) is

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$